MATH 632: ALGEBRAIC GEOMETRY II

These notes are the result of a collaboration involving the notes of several students in Algebraic Geometry II, taught Professor William Fulton at the University of Michigan. For suggestions and corrections of errors, email the organizer edeany@umich.edu.

Professor Fulton's office hours are Tuesday from 1pm to 2pm, Wednesday 3pm - 4pm, and Thursday 1pm - 2pm. Come at the beginning on Tuesdays or Thursdays or contact him ahead of time. Good times are also Tuesdays 2pm and onwards, Thursdays 2pm and onwards, and Fridays 4pm and onwards, but make sure to give advanced notice if you choose to come at these times.

1. Thursday, January 10th

632 (Algebraic Geometry II) will be a more conceptual and abstract course than 631 (Algebraic Geometry I). Professor Fulton makes an analogy with construction: 631 is to hand tools as 632 is to power tools. One danger, though, is that people think that, having mastered the 'power tools' that 632 offers, that they can do anything. That belief is mistaken. Just as one cannot build a house with just a bulldozer, one cannot be an Algebraic Geometer knowing only the more abstract and conceptual parts. So, Math 632 will be more abstract than 631, with more big conceptual ideas, but do not get the impression that it's a replacement for the 'hand tools' that you have learned in 631. They are equally important.

Common thinking about AG holds that it is a collection of facts, and that it doesn't matter how one learns any of these facts, so long as one knows them in the end. Under this perspective, working out the answers on one's own and looking up the answers on the internet are of equal value. One doesn't need to know the history of mathematics, go to seminars, or do exercises. Someone who has read Hartshorne is ready for research.

Compare this common thinking with analogous statements about music: one could say that music is a collection of notes, and that it doesn't matter how one learns the notes, so long as one knows them in the end. Under this perspective, learning to play a piano on one's own and listening to someone play it on the internet are of equal value. One doesn't need to know the history of music, go to concerts, or do exercises. Someone who knows their chords and scales is ready for Carnegie Hall.

The latter probably seems rediculous to you, but I think these are equally rediculous. Reading someone else's answers is about 1% as valuable as working out the solution on one's own. To be an algebraic geometer, one can't just hear about it. AG is a craft, and one must learn it by doing it.

After this discussion, we come to the subject of today's lecture. The development from math 631 to math 632 can be compared with a paradigm we see in the history of mathematics. Namely, it is the development mathematicians made from defining only embedded mathematical objects to making an intrinsic definition independent of any embedding. For instance, in the 19th century, algebraists did not have the notion of an abstract group. Instead, a group was deined as a certain subset of S_n (the symmetric group), of Gl_n (the general linear group), or of the automorphism group of a field. The situation is the same here: we know what an embedded algebraic variety is, but not one defined independently of any embedding yet. The varieties we know are affine (embedded in $\mathbb{A}^n$), projective (embedded in $\mathbb{P}^n$), or quasi-projective (the intersection of a closed subset and an open subset of $\mathbb{P}^n$). So our first goal is to define a notion of a variety independent of any embedding - abstract varieties.

In fact, it wasn't until the advent of modern mathematics that objects such as varieties were defined independently of any embedding. In 1828, Gauss showed that a certain property of surfaces in $\mathbb{R}^3$ is independent of their embedding - their curvature. In 1859, Riemann knew that M_g, the set of isomorphism classes of Riemann surfaces of genus g, itself forms a variety, the moduli space. M_0 consists of a single point, and M_1 is the one-dimensional family of eliptic curves. For larger genus, M_g has complex dimension $3g-3$. This space did not come as an embedded variety. Weyl gave a formal definition of a Riemann Surface.

This generalizes to a definition of an abstract manifold. We first define an atlas for a manifold, and then a manifold:

Definition 1. An atlas for consists of an open cover $\{U_\alpha\}_{\alpha\in A}$ of a topological space M, and a homeomorphism $\phi_\alpha : U_\alpha \to O_\alpha$ onto an open subset $O_\alpha \subset \mathbb{R}^n$ for each $\alpha \in A$ such that, for each α, β, the map $\phi_\beta \circ \phi_\alpha^{-1}$ is C^∞ (note that this is defined on $\phi_\alpha(U_\alpha \cap U_\beta)$).

We say two atlases $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$ and $\{(V_\beta, \phi_\beta)\}_{\beta\in B}$ are equivalent if, for each $\alpha \in A$ and each $\beta \in B$, the map $\psi_\beta \circ \phi_\alpha^{-1}$ is C^∞ (note that this is defined on $\phi_\alpha(U_\alpha \cap V_\beta)$), and similarly, $\phi_\alpha \circ \psi_\beta^{-1}$ is C^∞.

Definition 2. A C^∞ manifold is a topological space M with an equivalence class of atlases.

Often people add conditions, such as requiring M to be second countable or Hausdorff, but we aren't concerned with that here.

Definition 3. A morphism of C^∞ manifolds M and N is a continuous map $f : M \to N$ such that, for some (or any) atlases $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in I}$ of M and $\{(V_\beta, \psi_\beta)\}_{\beta\in B}$ of N, and for each $\alpha \in A$, and each $\beta \in B$, the map $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is C^∞ (note that this is defined on $\phi_\alpha(U_\alpha \cap f^{-1}(V_\beta))$).

Note that we can do the same for a real analytic or complex analytic manifold, replacing C^∞ with 'real analytic' or 'complex analytic' (and, in the case for a complex analytic manifold, we replace $\mathbb{R}^n$ with $\mathbb{C}^n$). Any of these classes will in fact form a category.

Exercise. *C^∞ manifolds and C^∞ maps of C^∞ manifolds form a category.*

Modern algebraic geometry grew up in the 1950s. Mathematicians in Paris were studying complex analytic spaces with singularities. In some sense, one could study these locally and glue them together, just as we we did for manifolds. However, mathematicians have found a better way.

In the air was Gelfand's idea that the ring $\mathbb{C}(X)$ of continuous $\mathbb{C}$-valued continuous functions on a compact hausdorff space X determines X. The maximal ideals of $\mathbb{C}(X)$ are then in correspondence with the points X. In fact, one can recover the topology X from $\mathbb{C}(X)$ when X is compact hausdorff. The idea here is that we should consider the ring of functions on the space, not the space alone.

Definition 4. Fix a field k. For a topological space U, we write $F(U,k)$ for the ring of k-valued functions on U. A k-ringed Cartan space (or simply Cartan space, with k implicit) consists of a topological space X and a sub-k-algebra $\mathcal{O}_X(U) \subset F(U,k)$ for each open subset U of X, such that the following two conditions are met:

(i) For any $V \subset U \subset X$ open, the restriction of any g in $\mathcal{O}_X(U)$ to a function on V is in $\mathcal{O}_X(V)$.
(ii) For any open covering $\{U_\alpha\}$ of $U \subset X$ open, and any $g \in F(U,k)$, g is in $\mathcal{O}_X(U)$ if and only if each restriction $g|_{U_\alpha}$ is in $\mathcal{O}_X(U_\alpha)$.

We might think of the elements of $\mathcal{O}_X(U)$ as those functions in $F(U,k)$ satisfying some nice property (the 'good' functions). (ii) says that this is a local property.

N.B. One may take k to be any commutative ring, but we are not interested in that level of generality at the moment.

Example 5. Let X be a topological space, and for each open set U of X, put $\mathcal{O}_X(U) = F(U,k)$. This forms a Cartan space.

Example 6. Let X be a C^∞ manifold, $k = \mathbb{R}$, and for each open set U of X, let $\mathcal{O}_X(U)$ be the set of C^∞ functions from U to $\mathbb{R}$. This forms a Cartan space. Similarly, we could take $\mathcal{O}_X(U)$ to be the ring of real or complex analytic functions on a real or complex manifold to get a Cartan space.

2. Tuesday, January 15th

Last lecture we defined a Cartan space with respect to a field k. We write $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ for Cartan spaces, which are supposed to have the same choice of k. We often write simply X and Y for $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$. Next we wish to define a notion of morphism of Cartan spaces:

Definition 7. A morphism from a Cartan space X to a Cartan space Y is a continuous mapping $f : X \to Y$ such that, for every open $U \subset Y$, the map $F(U,k) \to F(f^{-1}(U),k)$ taking g to $g \circ f$ takes $\mathcal{O}_Y(U)$ to $\mathcal{O}_X(f^{-1}(U))$.

Exercise. *For a fixed choice of k, Cartan spaces form a category. In particular, we have a notion of isomoprhism.*

Exercise. *Any open $V \subset \mathbb{R}^n$ is a Cartan space $k = \mathbb{R}$, with $\mathcal{O}_V(U) = \{C^\infty$ functions $U \to \mathbb{R}\}$, for any U open in V.*

Exercise. *If $(X, \mathcal{O}_X)$ is a Cartan space, and $X_0 \subset X$ is an open subset, then $(X_0, \mathcal{O}_{X_0})$ is a Cartan space, where we set $\mathcal{O}_{X_0}(U) = \mathcal{O}_X(U)$ for $U \subset X_0$ open. We write $\mathcal{O}_X|_{X_0}$ for $\mathcal{O}_{X_0}$*

The notion of a Cartan space allows for an alternative definition of a manifold:

Definition 8. A manifold is a Cartan space $(X, \mathcal{O}_X)$, with $k = \mathbb{R}$, that is locally isomoprhic to $(V, \mathcal{O}_V)$, where $V \subset \mathbb{R}^n$ is an open subspace of $\mathbb{R}^n$, made into a Cartan space as above. More precisely, we require that each point of X have a neighborhood U so that $(U, \mathcal{O}_X|_U)$ is isomorphic to $(V, \mathcal{O}_V)$ as Cartan spaces.

Exercise. *These two definitions are equivalent. If X is a C^∞ manifold under our old definition, then we define $\mathcal{O}_X(U)$ to be the set of C^∞ real-valued functions for each open subset U of X, giving $(X, \mathcal{O}_X)$ the structure of a Cartan space over $\mathbb{R}$. Conversely, suppose we have a Cartan space $(X, \mathcal{O}_X)$, as in definition 8. Take an open cover $\{U_\alpha\}_{\alpha \in A}$, where each U_α is isomorphic as a Cartan space to some Cartan space $(V_\alpha, \mathcal{O}_{V_\alpha})$, where $V_\alpha \subset \mathbb{R}^n$ is open, and $(V_\alpha, \mathcal{O}_{V_\alpha})$ gets its Cartan space structure from $\mathbb{R}^n$ as above. Show that this data gives an atlas, and that any two arising this way are equivalent.*

We can do the same for real analytic and complex analytic manifolds, replacing C^∞ with real analytic or copmlex analytic (and replacing $\mathbb{R}^n$ with $\mathbb{C}^n$ in the latter case). What mathematicians in the 1950's realized was that Cartan spaces are also a good way to deal with spaces with singularities:

Exercise. *Take $W \subset \mathbb{C}^n$ open, and $V \subset W$ closed, defined by a finite number of holomorphic functions. For $U \subset V$ open, we set $\mathcal{O}_V(U)$ to be the set of functions $f : U \to \mathbb{C}$ such that every point of U has a neighborhood of the form $U' \cap V$, with $U' \subset W$ open, and there is a holomorphic function F on U' so that $F = f$ on $U' \cap V$. This forms a Cartan space.*

Definition 9. A $\mathbb{C}$-analytic space is a Cartan space X such that, for every point $p \in X$, there is an open neighborhood U of p in X such that $(U, \mathcal{O}_X|_U) \cong (V, \mathcal{O}_V)$ for some Cartan space of the form above.

Our next goal is to define an algebraic variety. We would like to do this by 'gluing together affine varieties'. Let k be an algebraically closed field. We will start by making a closed algebraic set into a Cartan space.

Exercise. *This construction forms a Cartan space.*

Now we may form an algebraic variety by gluing spaces of this kind together:

Definition 10. An algebraic variety (over k) is a Cartan space $(X, \mathcal{O}_X)$ which is locally isomorphic to one of these. That is, there is an open cover $\{U_\alpha\}_{\alpha \in A}$ of X an closed subsets V_α of $\mathbb{A}^{n_\alpha}$, such that $(U_\alpha, \mathcal{O}_X|_{U_\alpha}) \cong (V_\alpha, \mathcal{O}_{V_\alpha})$ as Cartan spaces.

We also have a natural definition of morphisms on varieties:

Definition 11. A morphism $f : X \to Y$ of algebraic varieties is a morphism of X and Y as Cartan spaces.

Exercise. *These form a category.*

Claim 1. *If X is an algebraic variety, $U \subset X$ open, then U is an algebraic variety.*

Exercise. *If $X \subset \mathbb{A}^n$ is affine, closed, and $U \subset X$ is an open subset, then $\{X_f\}_{f \in k[X_1,\dots,X_n]}$ form a basis for the open sets of X, and each X_f is an affine variety. In fact $X_f \subset \mathbb{A}^{n+1}$.*

Exercise. *Let $X \subset \mathbb{A}^n$ be an algebraic set (i.e. X is a closed subset of $\mathbb{A}^n$). Then the canonical map $k[X_1,\dots,X_n]/I(X) \to \mathcal{O}_X(X)$ is an isomorphism.*

Proof. That this is injective is a tautology. This identifies A with a subset of $\mathcal{O}_X(X)$.

To show that this map is surjective, take $f \in \mathcal{O}_X(X)$. We aim to show that $f \in A$. The set $J = \{d \in A : d \cdot f \in A\}$ is an ideal. If $1 \in J$, then $f \in A$, as desired. Suppose for a contradiction that $1 \notin J$. Then $J \subset \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. $\mathfrak{m} = I(\{p\})$ for some point $p \in X$ by the Nullstellensatz. We know that there are $g, h \in A$ so $h(p) \neq 0$ and $f = g/h$ on some open neighborhood of p. We can multiply h by h' where $h'(p) \neq 0$, so that $(h'h) \cdot f = h'g$ on X. Now $h', h \in J$, while $h'h \notin \mathfrak{m}_p$, a contradiction. It follows that $1 \in J$, so that $f \in A$. □

N.B. the hard part of this proof seems to be coming up with the ideal J.

There is a similar statement for the so-called 'principal open subsets', which correspond to taking localizations:

Proposition 12. *For any $f \in A$, $f \neq 0$, $A_f \to \mathcal{O}_X(X_f)$ is an isomorphism.*

Definition 13. Let $V \subset \mathbb{A}^n$ be a closed set. To make V into a Cartan space, we give V the Zariski topology. For $U \subset V$ open, we set $\mathcal{O}_V(U)$ to be the set of functions from U to k such that, for each point $p \in U$, there is an open neighborhood U_p of p in $\mathbb{A}^n$, and $g, h \in k[X_1,\dots,X_n]$ with $h \neq 0$, such that $f = g/h$ on U_p.

Exercise. *Let $X \subset \mathbb{P}^n$ be a closed algebraic set. Let $\mathcal{O}_X(U)$ be the set of functions $f : U \to k$ such that every point $p \in U$ has a neighborhood so that $f = g/h$ on this neighborhood, where g and h are homogeneous polynomials in $k[X_0,\dots,X_n]$ of the same degree, with $h \neq 0$ in this neighborhood. This forms a Cartan space, and in fact it forms an algebraic variety. Define $U_i = \{z : z_i \neq 0\}$. For each $0 \leq i \leq n$, U_i forms an open subset of $\mathbb{P}^n$, and is isomorphic to $\mathbb{A}^n$ as a Cartan space. $\mathbb{P}^n = U_0 \cup \cdots \cup U_n$. For each $0 \leq i \leq n$, $X \cap U_i$ is isomorphic to a closed set $V_i \subset \mathbb{A}^n$.*

Open subsets of closed subsets of affine space and projective space also form Cartan spaces. These assertions are left as exercises:

Exercise. *Any $U \subset X \subset \mathbb{A}^n$ open is an algebraic variety (called quasi-affine).*

Exercise. *Any $U \subset X \subset \mathbb{P}^n$ open is an algebraic variety (called quasi-projective).*

Definition 14. A variety is called affine if it is isomorphic to some closed subset of $\mathbb{A}^n$, projective if it is isomorphic to some closed subset of $\mathbb{P}^n$, quasi-affine if it is isomorphic to some open subset of a closed subset of $\mathbb{A}^n$, and quasi-projective if it is isomorphic to some open subset of a closed subset of $\mathbb{P}^n$.

Example 15. M_g is quasi-projective, but the embedding is quite hard to write down. Often the explicit embedding does not matter.

3. Thursday, January 17th

Example 16. $\mathbb{A}^1 - \{0\}$ is an affine variety, as it is isomorphic to $V(xy-1) \subset \mathbb{A}^2$, and a closed subset of $\mathbb{A}^2$ is affine. Similarly, for $V \subset \mathbb{A}^n$ closed, V_f has a closed embedding into $\mathbb{A}^{n+1}$.

Example 17. $\mathbb{A}^2 - \{0\}$ is not affine. One can show that $\mathcal{O}_{\mathbb{A}^2-\{0\}}(\mathbb{A}^2 - \{0\}) = k[x,y]$, and that the embding $\mathbb{A}^2 - \{0\} \to \mathbb{A}^2$ induces an isomorphism of their rings of functions. Since there is a correspondence between affine varieties and reduced finitely generated k-algebras, the embedding $\mathbb{A}^2 - \{0\} \to \mathbb{A}^2$ should be an isomorphism if $\mathbb{A}^2 - \{0\}$ is affine. But it isn't, since this is not surjective as a set map. So $\mathbb{A}^2 - \{0\}$ cannot be affine.

About the correspondence between affine varieties and finitely generated, reduced k-algebras: a morphism $f : X \to Y$ of varieties induces a 'pullback' map $f^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$, a k-algebra map sending $g : Y \to k$ to $g \circ f : X \to k$. This gives a correspondence between affine varieties and finitely generated reduced k-algebras as follows:

Proposition 18. *If Y is affine, then any homomorphism of k-algebras $g : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ has $g = f^*$ for some $f : X \to Y$.*

Note that X need not be affine here. The proof of this is essentially an exercise. Take $Y \subset \mathbb{A}^n$, with coordinate functions $y_1, ..., y_n \in \mathcal{O}_Y(Y)$. The general idea is that any choice of $f_1, ..., f_n$ in $\mathcal{O}_X(X)$ for images of y_i gives n maps $f_1, ..., f_n : X \to \mathbb{A}$, and these induce a map $X \to \mathbb{A}^n$ into the product, whose image is contained in Y.

The correspondence is stating that a finitely generated reduced k-algebra 'looks just like' $\mathcal{O}_X(X)$ for an affine variety X. Such an algebra is isomorphic to one of the form $A[X_1, ..., X_n]/I$ for I a radical ideal. Formalizing this and combining it with the previous proposition gives us the following:

Claim 2. *The category of affine varieties is categorically equivalent to the opposite category of finitely generated reduced k-algebras.*

We turn to a discussion of what constructions can be made in the category of varieties. A simple one is disjoint union, which we leave as an exercise:

Exercise. *If X_α are algebraic varieties, then so is $\sqcup_\alpha X_\alpha$.*

Closed subsets of algebraic varieties also form algebraic varieties:

Definition 19. Let $(X, \mathcal{O}_X)$ be a Cartan space, and let $Z \subset X$ be a closed subset. We construct a Cartan space $(Z, \mathcal{O}_Z)$ as follows: for $U \subset Z$ open, we set $\mathcal{O}_Z(U)$ to be the set of functions $f : U \to k$ for which there exists an open neighborhood V of p in X and $F \in \mathcal{O}_X(V)$ with $F|_{U \cap V} = f$.

Proposition 20. *If $(X, \mathcal{O}_X)$ is an algebraic variety, and $Z \subset X$ is a closed subset, then $(Z, \mathcal{O}_Z)$ is an algebraic variety.*

We leave the proof as an exercise also. The basic idea is that we can assume $X \subset \mathbb{A}^n$ is closed, since this is a local question.

As it turns out that this proposition is special for varieties. Consider the following Cartan spaces:

(1) X is a discrete space, and $\mathcal{O}_X$ is k-valued functions on X.
(2) X is a topological space, and $\mathcal{O}_X$ is continuous functions on X.
(3) X is a topological manifold, and $\mathcal{O}_X$ is continuous functions.
(4) X is a C^∞ manifold, and $\mathcal{O}_X$ is the set of C^∞ functions on X.
(5) X is a real-analytic manifold, and $\mathcal{O}_X$ is the set of real-analytic functions on X.
(6) X is a complex-analytic, and $\mathcal{O}_X$ is the set of holomorphic functions on X.
(7) X is a $\mathbb{C}$-analytic space.
(8) X is an algebraic variety.

For which of these does the previous proposition hold? The answer is none, except for the first and the last. Under certain conditions, the proposition holds for some of the others. For instance, the proposition holds for the second if X is a normal space.

Similarly, if $Z \subset X$ is a locally closed subset of an algebraic variety, then we can make Z into a variety (in a unique way). To define $\mathcal{O}_Z$, we choose $U \subset X$ open so that $Z \subset U \subset X$ and define $\mathcal{O}_Z$ in the same manner as above. However, we must check that our choice is independent of the choice of U.

We can also form products in the category of varieties. We define them categorically, but note that it does not follow a priori that they exist.

Definition 21. If X and Y are algebraic varieties, then the product is a variety $X \times Y$ together with morphisms $p : X \times Y \to X$ and $q : X \times Y \to Y$, projections, such that, for any variety S with maps $f : S \to X$ and $g : S \to Y$, there is a unique map $(f,g) : S \to X \times Y$ such that $p \circ (f,g) = f$ and $q \circ (f,g) = g$.

Proposition 22. *If $X \times Y$ exists, it is unique up to canonical isomorphism.*

Proof. Take another variety P satisfying the universal property for products, with projection maps p' and q'. There is a unique map $f : P \to X \times Y$ such that $p \circ f = p'$ and $q \circ f = q'$. There is also a unique map $g : X \times Y \to P$ such that $p' \circ f = p$ and $q' \circ f = q$. Then $f \circ g : X \times Y \to X \times Y$ has $p \circ f \circ g = p$ and $q \circ f \circ g = q$, but we know that $1_{X \times Y}$ is the unique such map. So $f \circ g = 1_{X \times Y}$. Similarly, $g \circ f = 1_P$. This shows that f and g are inverse isomorphisms. □

Products do exist in the category of varieties, though they are not the same as products category of Cartan spaces, nor does this product have the product topology, as in the category of topological spaces. We mostly leave the proposition as an exercise:

Exercise. *Products exist in Algebraic Varieties.*

The idea of the proof is to start with the case where X and Y are affine. In this case, choose embeddings $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$ such that X and Y are closed subsets in $\mathbb{A}^m$ and $\mathbb{A}^n$ respectively. Then $X \times Y \subset \mathbb{A}^m \times \mathbb{A}^n = \mathbb{A}^{n+m}$, and $X \times Y$ becomes a closed subset of $\mathbb{A}^{m+n}$ in the Zariski topology, from which it gets a unique structure as an algebraic variety. $X \times Y \subset \mathbb{A}^{m+n}$ is the product of X and Y in the category of algebraic varieties, i.e. it satisfies the universal property explained above. Its coordinate ring is $\mathcal{O}_X(X) \otimes_k \mathcal{O}_Y(Y)$. For the general case, we glue together affine varieties.

Exercise. *There are canonical isomorphisms* $X \times Y \cong Y \times X$ *and* $(X \times Y) \times Z \cong X \times (Y \times Z)$.

Showing the second of these canonical ismorphisms is easier if we introduct a universal property for the product of 3 varieties (which is perfectly analogous to the one for two). We may similarly introduce the universal property of the product $\prod_{i=1}^n X_i$ of n varieties, and show that it exists in this category. In the category of varieties, we may not form infite products $\prod_{i=1}^\infty X_i$ in general.

Often by the word 'variety', one has more conditions in mind:

(1) Finite type: X is a finite union of affine open subsets.
(2) Irreducible: not a union of two proper closed subsets.
(3) Separated: the diagonal $\Delta_X \subset X \times X$ is closed. This is analogous to the hausdorff condition.

Here is a nonexample of a separated variety:

Example 23. Take $X_1 = X_2 = \mathbb{A}^n$, $U_1 = U_2 = \mathbb{A}^n - \{(0,0,...,0)\}$. Glue X_1, X_2 by the identity on U_1, U_2. The resulting variety is not separated.

Any finitely generated reduced k-algebra A, with k algebraically closed, is isomorphic to $\mathcal{O}_X(X)$ for some affine variety Specm(A), constructed from A, and unique up to canonical isomorphism. Moreover, a surjective map $k[X_1, ..., X_n] \to A$ gives an embedding Specm$(A) \to \mathbb{A}^n$. One idea of Grothendieck was to remove the requirement that A be reduced, finitely generated over k, or even a k-algebra at all. We can make the construction Spec(A) for any commutative ring, but all this comes later.

We have an analogous construction for projective varieties. Given $X \subset \mathbb{P}^n$ closed, $I(X) \subset k[X_0, ..., X_n]$ is a homogeneous ideal. $k[X_0, ..., X_n]/I(X)$ is a graded reduced k-algebra, equal to $k \oplus S_1 \oplus S_2 \oplus \cdots$, where S_i is consists of sums of polynomials whose terms have degree i. This is called the homogeneous coordinate ring of X. Conversely, given $S = \oplus_{d \geq 0} S_d$ a finitely generated graded reduced k-algebra, we want to construct an algebraic variety. We'll continue a discussion on this point in the next lecture.

4. Tuesday, January 22th

4.1. **Projm.** Let $S = \oplus_{d \geq 0} S_d$ be a finitely generated reduced graded k-algebra. That S is graded means that it decomposes as $\oplus_{d \geq 0} S_d$ as a k-module, and that the multiplication respects the grading ($S_d \cdot S_e \subset S_{d+e}$). Note that S is finitely generated by *graded* elements, since one can take homogeneous components of each one. Then S_0 is a subalgebra, and a finitely generated one since it is generated by the degree 0 parts of generators of S.

This algebra determines an affine variety, but it also determines a projective variety $\mathrm{Projm}(S)$, which we are currently interested in constructing. We wish to construct $\mathrm{Projm}(S)$ from S alone, without aid of an ambiant space. Our first case in introducing this construction is to suppose $S_0 = k$, and that S generated by $X_0, ..., X_n$ in S_1; the $X_0, ..., X_n$ will be considered part of the data of this construction. In this case we have $S = k[X_0, ..., X_n]/I$, where I is a homogeneous ideal and $I = \oplus_{d \geq 0} I_d$. $\mathrm{Projm}(S)$ will be the projective variety $V(I) \subset \mathbb{P}^n$. We say S is the homogeneous coordinate ring of $X = V(I)$. Note that this does not make S a ring of functions on $\mathrm{Projm}(S)$). For $F \in S_d$, $V(F) = \{x | F(x) = 0\} \subset X$ forms a closed subvariety. $X_F = X - V(F) = \{x | F(x) \neq 0\}$ forms an open subvariety of $\mathbb{P}^n$, and in fact it forms an affine variety.

We proceed to show this for a homogeneous F homogeneous in $k[X_0, ..., X_n]$ of degree d. (In general, X_F will be a closed subset of this, which will then be affine since a closed subvariety of an affine variety is affine). First consider the case where $d = 1$, so that $F = \sum_{i=0}^n a_i X_i$. Applying some linear automorphism of $\mathbb{P}^n$ whose corresponding map $k[X_0, ..., X_n]$ sends F to one of X_i, we may assume that X_F is one of the standard open affine sets, so that it is affine.

Now we use the veronese embedding $v_d : \mathbb{P}^n \hookrightarrow \mathbb{P}^N$, where $N = \binom{n+d}{n} - 1$ to reduce to the case where F has degree 1. $v_d(\mathbb{P}^n)$ sends X_F to the complement of a hyperplane in $\mathbb{P}^N$, and this reduces the claim to the first case. The details are left as an exercise.

Now that we have seen that X_F is an affine variety, we ought to be able to write down its coordinate ring. In fact, the coordinate ring is $S_{(F)} = \{G/F^m \in S_F : G \in S_{md}\}$. We say $X_F = \mathrm{Specm}(S_{(F)})$. Having constructed X_F for homogeneous polynomials F, we may construct X by gluing the affine varieties X_F. One can show that $X_F \cap X_G = X_{FG}$, so that one constructs X by gluing affine varieties of the form X_F by gluing together at the affine varieties X_{FG}. The question thus becomes, when do $X_{F_1}, ..., X_{F_r}$ cover X for homogeneous polynomials F_i, so that we may proceed with this gluing construction? We have the following theorem to answer this question:

Proposition 24. *The following are equivalent:*

(1) $X_{F_1} \cup \cdots \cup X_{F_r} = X$
(2) $V(F_1, ..., F_r) = \varnothing$.
(3) $X_i^N \in (F_1, ..., F_r)$ *some* $N \geq 0$, *and for all* $0 \leq i \leq n$
(4) $(S_+)^N \subset (F_1, ..., F_r)$, *where* $S_+ = \oplus_{d>0} S_d$.
(5) $S_d \subset (F_1, ..., F_r)$ *some* $d > 0$.

In $X \subset \mathbb{P}^n$, points correspond to prime ideals, not maximal ideals. These ideals $\mathfrak{m} \subset S = k[X_0, ..., X_n]/I \leftrightarrow k[X_0, ..., X_n]$ (equivalently, ideals $\mathfrak{m} \subset k[X_0, ..., X_n]$ containing I) may be characterized as those which are maximal among those which do not contain S^+. For instance, the point $[1 : 0 : \cdots : 0] \in \mathbb{P}^n$ corresponds to the ideal

$\mathfrak{m} = (X_1, .., X_n)$. Then $k[X_0, ..., X_n]/\mathfrak{m} = k[x_0]$, illustrating that $\mathfrak{m}$ does not correspond to a maximal ideal.

Until now, we have assume that $S_0 = k$, and that S is a finitely generated reduced graded k-algebra. Our second, broader case, is to take S to be finitely generated by elements in S_1 over S_0 (we are no longer supposing that $S_0 = k$, but instead that $k \subset S_0$). Hence $S = S_0[X_0, ..., X_n]/I$. Here, we get $X = \text{Projm}\ (S) = V(I) \subset \text{Specm}(S_0) \times \mathbb{P}^n$ (this is not equality). $\text{Specm}(S_0)$ is an affine variety. We do this again by gluing together the affine subvarieties $X_F \subset X$; the case proceeds just as before, except that now $X_F \subset \text{Specm}(S_0) \times \mathbb{P}^n$, where $X_F = \text{Specm}(S_{(F)})$. Note that here we have a morphism $X \to \text{Specm}(S_0)$.

For the general case, we declare the points of $\text{Projm}(S)$ to be homogeneous ideals $\mathfrak{m}$ maximal among those which do not contain S_+. We have a Zariski topology on these, where $V(F) = \{\mathfrak{m} | F \in \mathfrak{m}\}$ and X_F is its complement. The sets of the form X_F form a basis for this topology. Each X_F has structure of affine variety, with coordinate ring $S_{(F)}$. We define X by gluing these (note $X_F \cap X_G = X_{FG}$). Explicitly, $f : U \to k$ is in $\mathcal{O}_X(U)$ when, for some F with $X_F \subset U$, $f|_{X_F}$ is given by a restriction of some function of the form $G/F^m \in S_{(F)}$. This concludes our construction of $\text{Projm}(S)$.

Example 25. Let Y be an affine variety with coordinate ring $A = \mathcal{O}_X(X)$. Let $Z \subset Y$ be a closed subvariety, and write $I = I(Z)$, $\mathcal{O}_Z(Z) = A/I$. Take $S = \oplus_{d \geq 0} I^d = A \oplus I \oplus I^2 \oplus \cdots$, making this into a k-algebra by the embedding $I^d I^e \subset I^{d+e}$. S is graded, reduced, and finitely generated over A by finitely many elments in S_1. Putting $X = \text{Projm}(S)$, we have a map $\text{Projm}(S) \to Y$, called the blowup of Y along Z.

Definition 26. We write $X = B|_Z Y$ above, and call this construction the blowup of Y along Z. If $X_0, ..., X_n$ generate I, then $\text{Bl}_Z(Y) \subset Y \times \mathbb{P}^n$.

4.2. **Toric Varieties.** Next we delve into a class of varieties called 'toric varieties'. These give one an enormous collection of examples to work with, and serve as a testing ground for conjectures

Definition 27. For a group G and a field k, recall the construction of the *group algebra* $k[G]$, whose elements are formal sums of basis elements χ^g for $g \in G$. The product on $k[G] \cong \oplus_{g \in G} k \chi^g$ is determined by specifying that $\chi^g \chi^h = \chi^{gh}$.

Example 28. Let M be a lattice (a finitely generated free abelian group, isomorphic to $\mathbb{Z}^n$). M can be identified as a subset of $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$, which can be identified as a subset of $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\Delta \subset M_{\mathbb{R}}$ be the convex hull of a finite number of vectors $v_1, ..., v_n$ in $M_{\mathbb{Q}}$, so that $\Delta = \{\sum r_i v_i : r_i \geq 0, \sum r_i = 1\}$. Δ is called a rational convex polytope. M has a corresponding group algebra $k[M]$. If a basis $e_1, ..., e_n$ is chosen for M, we want $X_i := X^{e_i}$, so for $u = (u_1, ..., u_n) \in \mathbb{Z}^n$, we write $\chi^u = X_1^{u_1} \cdots X_n^{u_n}$.

We wish to construct a graded algebra S from $k[M]$, which will be a subset of $k[M]$, and which will be an example of a toric variety. To do this, we set $S_0 = k$, and we set S_d to have as a basis the set of χ^u for $u \in M$ such that $(1/d)u \in \Delta$. Then $S_d S_e \subset S_{d+e}$, since for $u/d \in \Delta$ and $v/e \in \Delta$, $\frac{u+v}{d+e} = \frac{d}{d+e}\frac{u}{d} + \frac{e}{d+e}\frac{v}{e}$, so that, by convexity, $(u+v)/(d+e) \in \Delta$. This makes S is a graded ring, which is automatically reduced as it is a subset of the domain $k[M]$. It is a fact, related to a result called Gordan's lemma, that S is finitely generated, and with this we get a variety $X_\Delta = \text{Projm}(S)$, our first example of a toric variety.

5. Thursday, January 24th

Recall the setting of last lecture: we have a lattice M, and $M \subset M_\mathbb{Q} \subset M_\mathbb{R}$. Note that we have not chosen a basis for M, even though one exists. Take $C \subset M_\mathbb{R}$ a rational convex cone. That is, $C = \{\sum r_i v_i | r_i \geq 0\}$ for some $v_1, ..., v_m \in M_\mathbb{Q}$. In the last lecture, we mentioned a lemma, 'Gordan's lemma', which we now state and prove:

Lemma 29 (Gordan's Lemma)**.** *$C \cap M$ is a finitely generated subsemigroup of M.*

The proof of this is clever, but short. Someone asked why we need the reals, which is a good question; here is where they come into the picture.

Proof. (Sketch) Take generators $v_1, ..., v_m$ in M. Take $K = \{\sum r_i v_i : 0 \leq r_i \leq 1\}$. K is compact in $M_\mathbb{R} \cong \mathbb{R}^n$. So $K \cap M$ is a compact and discrete, and therefore finite. We claim that $K \cap M$ generates $C \cap M$, and this is an easy exercise. □

With M as above, take a rational convex polytope $\Delta = \{(r_i, v_i) : r_i \geq 0, \sum r_i = 1\} \subset M_\mathbb{R}$, where $(v_1, ..., v_m) \in M_\mathbb{Q}$. Set $S = \oplus S_d$, $S_0 = k$, where S_d has a basis χ^u for $u \in M$ such that $\frac{1}{d}u \in \Delta$. The convexity of Δ implies that this is a graded k-algebra. $S \subset k[M]$. The latter is a domain, so that S is, but what is not so obvious is that S is finitely generated. If it is, then we can form $\text{Projm}(S)$.

Proposition 30. *S is finitely generated.*

Proof. Consider $M \times \mathbb{Z}$. This is a lattice of rank $\text{rank}(M) + 1$. Let C be the cone in $(M \times \mathbb{Z})_\mathbb{R} = M_\mathbb{R} \times \mathbb{R}$ generated by $\Delta \times \{1\}$. The claim, which we leave as an exercise, is that $S = k[C \cap (M \times \mathbb{Z})]$. For $u \in S_d$ have $(u, d) \in C \cap (M \times \mathbb{Z})$. Gordan's lemma finishes the proof. □

We can now construct the projective variety $\text{Projm}(S)$. It can be shown that, for $C \subset M_\mathbb{R}$ a rational polyhedral cone, $\text{Specm}(k[C \cap M])$ is an affine toric variety.

Definition 31. $\text{Projm}(S)$ is the toric variety of Δ.

These constructions give an interesting and rich set of examples to work with.

Example 32. Let $M = \mathbb{Z}^n$ and $\Delta = \langle e_0, ..., e_n \rangle$. Then the projective toric variety of Δ is $\mathbb{P}^n$.

Example 33. Let $M = \mathbb{Z}^{n+1}/\mathbb{Z}(1, ..., 1)$. This is isomorphic to $\mathbb{Z}^n$, but not equal to it. Take $\Delta = \langle \overline{e_0}, ..., \overline{e_n} \rangle$, where $\overline{e_i}$ is the image of e_i in M. Put $S = k[X_0, ..., X_n]$, so that $\text{Projm}(S) = \mathbb{P}^n$.

Specm($k[M]$) = T is a torus isomorphic to $\mathbb{G}_m^n = (k^*)^n$. In fact, T is an algebraic group, and acts on these varieties making them into 'algebraic group action objects'. In general, an algebraic group is a variety X, with a map of varieties from $X \times X$ to X, and a map from $\mathbb{A}^1$ to X, with certain coherence conditions corresponding to associativity and the unit laws. One can similarly define a 'group action object'.

5.1. **Germs.** For $(X, \mathcal{O}_X)$ a Cartan space, and any $p \in X$, we have a k-algebra $\mathcal{O}_p(X)$ of germs of functions at p. This is the data of $(X, \mathcal{O}_X)$ 'arbitrarily near' the point p. Any function in a neighborhood of a point should define a germ (an element in $\mathcal{O}_p(X)$, but some functions define the same germ. Two functions $f : U \to k$ and $g : V \to k$ defined on open neighborhoods of p are the same in $\mathcal{O}_p(X)$ when there is an open set $W \subset U \cap W$ containing p, such that $f|_W = g|_W$. In modern terminology, we say $\mathcal{O}_p(X)$ is the direct limit $\lim_{p \in U} \mathcal{O}_X(U)$.

Definition 34. $(X, \mathcal{O}_X)$ is a locally k-ringed Cartan space if every $\mathcal{O}_p(X)$ is a local ring with maximal ideal $\mathfrak{m}_p(X) = \{f | f(0) = 0\}$. Then $\mathcal{O}_p(X)/\mathfrak{m}_p(X) \cong k$.

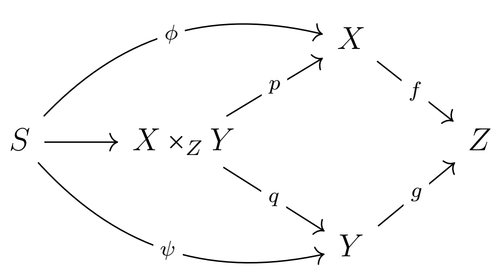
All our examples so far have satisfied this, and indeed all varities are locally k-ringed Cartan spaces. But here is a nonexample:

Example 35. Take $X = \mathbb{A}^n$ with the Zariski topology. Define $\mathcal{O}_X(U) = k[X_1, ..., X_n]$ for all $U \neq \varnothing$.

Note that this is a Cartan space, but not an algebraic variety.

5.2. **Fiber Products.** Here is some temporary notation: for a morphism $f : X \to Y$ of algebraic varieties, and a closed subset $Z \subset Y$ (respectively, locally closed), we write $f^{-1}(Z)$ for the closed subvariety of X induced by the preimage (respectively, locally closed subvariety). We will use this to make a more interesting construction, which makes sense in just about any setting.

Definition 36. Given two morphisms $f : X \to Z$, and $g : Y \to Z$, morphisms of algebraic varieties, there is a *fiber product* $X \times_Z Y$ is a variety with $p : X \times_Z Y \to X$ and $q : X \times_Z Y \to Y$, such that $f \circ p = g \circ q$, and for any variety S with $\phi : P \to X$ and $\psi : Q \to Y$ such that $f \circ \phi = g \circ \psi$, there is a unique map $(\phi, \psi) : P \to X \times_Z Y$ such that the following diagram commutes:



This is unique up to canonical isomorphism, if it exists. In fact, it is an exercise that $X \times_Z Y$ is $(f \times g)^{-1}(\Delta_Z)$, which fits into the following diagram:

$$\begin{array}{ccc} X \times Y & \xrightarrow{f\times g} & Z \times Z \\ \uparrow & & \uparrow \\ (f \times g)^{-1}(\Delta_Z) & \longrightarrow & \Delta_Z \end{array}$$

Example 37. Let $f : \mathbb{A}^1 \to \mathbb{A}^1$ be the regular map sending z to z^3. Then $f^{-1}(0) = 0$. It seems like the first zero should have a multiplicity encoded into our model of what is going on- intuitively, they are 'not the same 0'. This is where schemes come in, and it is why our current definition is only temporary.

5.3. **Vector Bundles.**

Definition 38. Lext X be a variety. A *family of vector spaces* over X is a variety E with a morphism $\pi : E \to X$, together with the structure of a finite-dimensional vector space on each fiber, $E_x = \pi^{-1}(x)$. We often do not include π explicitly when introducing a family of vector spaces. One writes π_E if more than one family of vector spaces is considered.

Definition 39. A morphism $f : E \to F$ of two such varieties over X is a morphism of varieties $f : E \to F$ so that

(1) the following diagram commutes

$$\begin{array}{ccccc} E & & \xrightarrow{\quad f \quad} & & F \\ & \searrow_{\pi_E} & & {}_{\pi_F}\swarrow & \\ & & X & & \end{array}$$

(2) the restriction of f to E_x gives a linear map from E_x to F_x for each $x \in X$.

Families of vector spaces then form a category.

Example 40. The trivial rank n bundle on X is the projection map $E = X \times k^n \to X$.

Example 41. If $E \to X$ is a family of vector spaces X, and $U \subset X$ is an open set, or a locally closed set, then $E|_U = \pi^{-1}(U) \to U$ is a family of vector spaces over U.

Example 42. Let $\pi : E \to X$ be a family of vector spaces is a vector bundle if it is locally trivial: every point has a neighborhood U with an isomorphism $E|_U \to U \times k^n$ of families of vector spaces over U. Note that n may depend on the point p, but it is constant on connected components of X.

6. Tuesday, January 29th

Written by Anna Brosowsky.

6.1. **More about Vector Bundles.** Let V be a finite dimensional vector space over k. Then V is an affine algebraic variety. We can see this in two ways:

(1) Choose a basis and get an isomorphism $V \cong k^n$. But then we have to check that this structure is independent of basis.

(2) Alternatively, view $V = \operatorname{Specm}(\operatorname{Sym}^\bullet(V^*))$. Why V^*? We want to be looking at function rings, and $\operatorname{Sym}^1(V^*)$ gives us linear functions on V. Choosing $V \cong k^n$ gives $V^* \cong k^n$ and $\operatorname{Sym}^\bullet(V^*) \cong k[x_1, \dots, x_n]$.

What even is a vector space basis? Again, there are two interpretations. Either a choice of isomorphism with k^n (i.e., ordered) or a set of vectors which uniquely generate each vector in V (i.e., unordered). So if V has basis $\{e_\alpha\}_{\alpha\in\mathcal{A}}$ and W has basis $\{f_\beta\}_{\beta\in\mathcal{B}}$ then a linear transformation $L : V \to W$ is given by a matrix $L(e_\alpha) = \sum a_{\beta\alpha} f_\beta$. Note there's not a "first" row here, and it's not even an $n \times n$ matrix, since everything is unordered. Instead, we'll say it's a $\mathcal{B} \times \mathcal{A}$ matrix, and L takes x with coordinates $\{x_\alpha\}$ to y with coordinates $\{y_\beta\}$, $y_\beta = \sum a_{\beta\alpha} x_\alpha$. The tensor product $V \otimes W$ has a basis $\{e_\alpha \otimes f_\beta\}_{\alpha\in\mathcal{A},\beta\in\mathcal{B}}$. Recall from last class

Definition 43. A vector bundle E on X (both algebraic varieties) is E with a map of varieties $\pi_E : E \to X$ a family of vector spaces which are locally trivial, i.e. each point has a Zariski open neighborhood U such that

$$\begin{array}{ccc} E|_U & \xrightarrow{\sim} & U \times V \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

where this is an isomorphism in the category of families of vector spaces, and V is a finite dimensional vector space.

Note that this construction also works if X and E are in the category of topological spaces, topological manifolds, C^∞ manifolds, $\mathbb{R}$ or $\mathbb{C}$ analytic manifolds, etc. But it doesn't work for Cartan spaces generally since we don't have a product there. Often, we'll take $V = k^n$ where n is locally constant.

We want to be able to form vector bundles by gluing, but we'll need some extra conditions on our maps to respect the extra structure. So to start, we'll look at a vector bundle and see what the transition functions must look like. Choose an open covering $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of X with local isomorphisms φ_α,

$$\begin{array}{ccc} E|_{U_\alpha} & \xrightarrow{\varphi_\alpha} & U_\alpha \times k^n \\ & \searrow \quad \swarrow & \\ & U_\alpha & \end{array}$$

such that on the intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$, we have a commutative diagram

$$\begin{array}{ccc} E|_{U_{\alpha\beta}} & \xrightarrow{\varphi_\alpha} & U_{\alpha\beta} \times k^n \\ \downarrow{\scriptstyle \varphi_\beta} & & \\ U_{\alpha\beta} \times k^n & & \end{array}$$

By composing, we have an isomorphism $\varphi_{\beta\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1}$ which is of the form

$$\varphi_{\beta\alpha}(x, v) = (x, \theta_{\beta\alpha}(x)(v))$$

with $\theta_{\beta\alpha}(x) \in \mathrm{GL}_n(k)$. In fact, $\theta_{\beta\alpha} : U_{\alpha\beta} \to \mathrm{GL}_n(k)$ is a morphism such that

(1) $\theta_{\alpha\alpha}(x) = I = 1$ in $\mathrm{GL}_n(k)$ for all $x \in U_{\alpha\alpha} = U_\alpha$;
(2) $\theta_{\gamma\alpha} = \theta_{\gamma\beta} \bullet \theta_{\beta\alpha}$ on $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$, i.e. we do pointwise matrix multiplication $\theta_{\gamma\alpha}(x) = \theta_{\gamma\beta}(x) \cdot \theta_{\beta\alpha}(x)$ in $\mathrm{GL}_n(k)$.

Conversely, given $X = \bigcup_{\alpha\in\mathcal{A}} U_\alpha$ an open cover and such $\theta_{\beta\alpha} : U_{\alpha\beta} \to \mathrm{GL}_n(k)$, we get a vector bundle E on X with isomorphisms $E|_{U_\alpha} \cong U_\alpha \times k^n$ and these transitions.

Vector bundles were (and can be) described by such data. But then we have to say when two such data give isomorphic vector bundles.

Exercise. $E \times_X E = \{(v,w) \mid \pi_E(v) = \pi_E(w)\} \xrightarrow{+} E$ *is a morphism of algebraic varieties. So are* $k \times E \xrightarrow{\cdot} E$, $X \xrightarrow{0} E$, *and* $E \xrightarrow{-} E$.

6.2. **Constructions with Vector Bundles.**

Example 44. If E is a vector bundle on X, $Z \subset X$ locally closed, then $E|_Z$ is as well, with structure given by

$$\begin{array}{ccc} E|_Z = \pi^{-1}(Z) & \hookrightarrow & E \\ \downarrow & & \downarrow \\ Z & \hookrightarrow & X \end{array}$$

Example 45. If $X \subset \mathbb{A}^m$ is a closed algebraic set, a point $p \in X$ is *nonsingular* if there are $f_1, \ldots, f_d \in \mathcal{O}_{\mathbb{A}^m}(U)$ for U a neighborhood of p such that $X \cap U = V(f_1, \ldots, f_d)$ and $[\frac{\partial f_i}{\partial x_j}(p)]$ the Jacobian matrix has rank d. Note that this is a $d \times n$ matrix, so is maximal rank, and that this condition must then hold on some neighborhood $U' \subset U$ as well.

Fact: these $f_1, \ldots, f_d$ then generate the ideal of X in some affine open neighborhood of p. For $x \in U'$, we have

$$T_x X = \{(v_1, \ldots, v_m) \in k^m \mid \sum_j \frac{\partial f_i}{\partial x_j}(x) v_j = 0, \quad 1 \le i \le d\}.$$

Alternatively, we also have that $\sum_j \frac{\partial f}{\partial x_j}(x) v_j = 0$ for all f in the ideal of X in U'. This has dimension $m - d = \dim(X \cap U')$.

Exercise. *If X is nonsingular at all points, this $TX = \bigcup_{x\in X}\{x\} \times T_x X$ forms a vector bundle over X, the* tangent bundle.

If E and F are both vector bundles on X, we have several more constructions.

Direct Sum: This $E \oplus F$ should satisfy the universal property for direct sums, and in fact the total space of $E \oplus F$ is $E \times_X F \subset E \times F$, the product a product as algebraic varieties. But actually $E \oplus F$ and $E \times F$ (with the product a product of vector bundles now) are the same!

Tensor Product: This should satisfy the usual universal property for tensor products:

$$\begin{array}{ccc} E \times F & \xrightarrow{\text{bilinear}} & G \\ {\scriptstyle\text{bilinear}}\downarrow & \overset{!}{\nearrow} & \\ E \otimes F & & \end{array}$$

where the product is a product in the category of vector bundles. If it exists it's unique, and the construction is to take $\{U_\alpha\}$ with trivial $E|_{U_\alpha} \cong U_\alpha \times k^m$, $F|_{U_\alpha} \cong U_\alpha \times k^n$ and glue the $U_\alpha \times (k^m \otimes k^n)$'s.

Exterior Product: $\bigwedge^k E = \mathrm{Alt}^k E$ is universal for alternating maps.

Symmetric Product: $\mathrm{Sym}^k E = S^k E$ is universal for symmetric maps.

Duals: E^* is a vector bundle on X, with transition functions $(\theta_{\beta\alpha}^{-1})^T$.

Hom Bundle: $\mathrm{Hom}(E, F)$ is locally $U \times \mathrm{Lin}(k^m, k^n)$.

Exercise. *Show that $E^* = Hom(E, \mathbb{1}_X)$ where $\mathbb{1}_X$ is the trivial bundle $X \times k \to X$. There is also a canonical isomorphism $E^* \otimes F \to Hom(E, F)$.*

As with vector spaces, there is a canonical isomorphism $\bigwedge^k(E^*) \to \left(\bigwedge^k E\right)^*$. But in small characteristic, it is *not* true that $S^k(E^*) \cong (S^k E)^*$.

Kernel: If $\varphi : E \to F$ is a map of vector bundles and φ is surjective then $\ker(\varphi)$ is a vector bundle of rank rk E − rk F. Locally, we have $U \times k^n \twoheadrightarrow U \times k^m$. The idea is to take a basis of k^m, take elements of k^n mapping onto each of them, and extend to a basis of k^n. Then can choose trivializations such that $U \times k^m \times k^r \to U \times k^m$ by projection, so the kernel is $U \times k^r$.

Cokernel: If $\varphi : E \to F$ is injective, then $\mathrm{Cok}(\varphi) = \mathrm{Coker}(\varphi)$ is a vector bundle.

For a general map φ, all we get is a family of vector spaces, but no bundle. The issue is that if the rank is changing, there's no way to put them together into a bundle. But if φ has (locally) constant rank, then $\ker\varphi$, $\mathrm{Im}\,\varphi$, and $\mathrm{coker}\,\varphi$ are all vector bundles. We get the following commutative diagram of exact sequences.

$$\begin{array}{ccccccccc} 0 \longrightarrow \ker\varphi \longrightarrow & E & & \xrightarrow{\quad\varphi\quad} & & F & \longrightarrow \mathrm{coker}\,\varphi \longrightarrow 0 \\ & & \searrow & & \nearrow & & \\ & & & \mathrm{Im}\,\varphi & & & \\ & & \nearrow & & \searrow & & \\ & 0 & & & & 0 & \end{array}$$

7. Tuesday, February 5th

Example 46. Let X be the variety $\mathbb{P}^n$, and U_i be the open set where the i^{th} coordinate x_i does not vanish. Define the transition functions from U_i to U_j to be the following.

$$\mathcal{O}_{j,i}([x_0 : \cdots : x_n]) = \frac{x_i}{x_j}$$

It's not hard to check that these transition functions satisfy the cocycle condition, and hence define a line bundle on X. This line bundle is denoted by $O(1)$. If the transition functions are $\left(\frac{x_i}{x_j}\right)^d$, then the corresponding line bundle is called $O(d)$.

It turns out that all of these line bundles are non-isomorphic, and every line bundle on $\mathbb{P}^n$ is isomorphic to some $O(d)$.

There is a more geometric way of constructing a line bundle on $\mathbb{P}^n$. Consider the rank $n+1$ trivial bundle over $\mathbb{P}^n$, and let S be the sub-bundle such that the fibre over a point $p \in \mathbb{P}^n$ is the line $l \subset k^{n+1}$, where the line l is the line that goes to p in the quotient map $k^{n+1} \smallsetminus \{0\} \to \mathbb{P}^n$. To check that this is a line bundle, we just need to check that it's locally trivial. On one of the open sets U_i, the local trivialization is given by the following map.

$$\varphi : U_i \times k \to S\big|_{U_i}$$
$$\varphi : ([x_0 : \cdots : x_n], t) \mapsto t\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right)$$

This shows that S is indeed a line bundle. It is often called the *tautological line bundle*. If one computes the transition functions of S, it turns out that they're exactly the transition functions of $O(-1)$, which means S is isomorphic to $O(-1)$.

Definition 47 (Section of a vector bundle)**.** A section of a vector bundle E over X is a morphism S from X to E such that $\pi_E \circ S$ is the identity map on X.

We will denote the set of sections of E by $\Gamma(X, E)$. It's not too hard to see that $\Gamma(X, E)$ is a vector space. In fact, they're more: for any open subset $U \subset X$, $\Gamma(U, E)$ is an $\mathcal{O}_X(U)$-module.

Locally, sections of vector bundles look particularly nice. In a locally trivial neighbourhood U, a section s of a rank n vector bundle looks like a regular map from U to k^n. Conversely, given such a function s_α from each locally trivial neighbourhood U_α to k^n, these maps can be glued together to get a section, if they satisfy the following condition for all U_α and U_β.

$$s_\beta(x) = \mathcal{O}_{\beta,\alpha}(x) \cdot s_\alpha(x)$$

In particular, we get a collection of sections of $O(d)$ for $d \geq 0$, as follows:

Example 48. On $\mathbb{P}^n$, any homogeneous polynomial F of degree d defines a section of the line bundle $O(d)$. The section is obtained by gluing together the locally defined maps $s_i = \frac{F}{x_i^d}$, by the transition maps of $O(d)$.

In fact, something much stronger is true.

Proposition 49. *For every section s of $O(d)$, there exists a homogeneous polynomial F of degree d such that in the locally trivial neighbourhood U_i, s is given by $\frac{F}{x_i^d}$.*

Proof. Consider what s looks like in two locally trivial neighbourhoods U_i and U_j. Since both those open sets are isomorphic to $\mathbb{A}^n$, regular functions from these sets to k^n will be polynomials. We then homogenize them with respect to the coordinates x_i (in U_i), and x_j (in U_j) to get the following equality.

$$\frac{F_j}{x_j^m} = \left(\frac{x_i}{x_j}\right)^d \frac{F_i}{x_i^m}$$

But since $k[x_0, \ldots, x_n]$ is a UFD, each F_i is equal to $x_i^{m-d}F$ for some homogeneous polynomial F. That is the polynomial we want. □

Now let X be any closed subset of $\mathbb{P}^n$, and let S be its homogeneous coordinate ring. What we've done so far shows that each element $s \in S$ gives us a unique section of E, i.e. the map from S to $\Gamma\left(X, O(d)\big|_X\right)$ is injective. For some X, e.g. $X = \mathbb{P}^n$, the map is also surjective, as the proposition we just proved shows.

Next, we investigate how the sheaf of modules $\Gamma(X, L)$ is generated for any line bundle L.

Exercise. *Let L be a line bundle on X, U an open subset of X, and s a section that vanishes nowhere in U. Show that for any for any other section $t \in \Gamma(U, L)$, there exists a regular function f on U, such that $t = f \cdot s$, i.e. the sheaf of modules $\Gamma(U, L)$ is generated by s over $\mathcal{O}_X(U)$.*

In fact, this exercise can be generalized further.

Lemma 50. *Let L be a line bundle on X. Assume we have sections $\{s_0, \ldots, s_n\}$ of $\Gamma(X, L)$ such that for any $x \in X$, there exists some sections s_i which doesn't vanish at x. Consider a map to $\mathbb{P}^n$ given by these sections.*

$$\varphi : X \to \mathbb{P}^n$$
$$\varphi : x \mapsto [s_0(x) : \cdots : s_n(x)]$$

Then there exists a canonical isomorphism between L and the pullback line bundle $\phi^*(O(1))$.

8. Thursday, February 7th

Written by Yin Hang.

8.1. Projective bundles and Grassmannian bundles.

Definition 51. Let $E \to X$ be a vector bundle of rank n, then the associated projective bundle $\mathbb{P}(E) \to X$ is defined by gluing local coordinates. If $\{E|_{U_\alpha} \cong U_\alpha \times k^n\}$ is a trivialization of E, then $\{\mathbb{P}(E)|_{U_\alpha} \cong U_\alpha \times \mathbb{P}^{n-1}\}$ is a trivialization of $\mathbb{P}(E)$. The transition map is induced from $U_{\alpha\beta} \times k^n \xrightarrow{\theta_{\beta\alpha}} U_{\beta\alpha} \times k^n$ to $U_{\alpha\beta} \times \mathbb{P}^{n-1} \xrightarrow{\bar{\theta}_{\beta\alpha}} U_{\beta\alpha} \times \mathbb{P}^{n-1}$.

There is an interesting line bundle on $\mathbb{P}(E)$. If $E \xrightarrow{p} X$ is a vector bundle, the tautological bundle of the associated bundle $\mathbb{P}(E)$ is the sub-bundle $S \subseteq p^*(E)$ defined by $x \times [l] \times l \subseteq U_\alpha \times \mathbb{P}^{n-1} \times k^n$, where $x \in X$, $[l] \in \mathbb{P}^{n-1}$ is the line defining the point, $U_\alpha \times \mathbb{P}^{n-1}$ is the trivialization of $\mathbb{P}(E)$, k^n is the line bundle corresponding to the pull back of E. Since l is a line in k^n, it naturally defines a line sub-bundle. (Check that this is locally trivial.)

We are interested in morphisms to $\mathbb{P}(E)$.

Claim: For any algebraic variety Y, $\mathrm{Hom}(Y, \mathbb{P}(E)) \cong \{f : Y \to X$, together with a line sub-bundle $L \subseteq f^*(E)\}$

Sketch of the proof: Given a morphism $g : Y \to \mathbb{P}(E)$, we can form $f = \pi \circ g$

$$\begin{array}{ccc} & & \mathbb{P}(E) \\ & \overset{g}{\nearrow} & \downarrow p \\ Y & \xrightarrow{f} & X \end{array}$$

Let $L = g*(S)$, which is a line sub-bundle of $f*(E)$ since $S \subseteq p^*(E)$ implies $g^*(S) \subseteq g^*p^*(E) = f^*(E)$.
Conversely, given $f : Y \to X$ and a line sub-bundle $L \subseteq f^*(E)$, we have

$$\begin{array}{ccc} L & \longrightarrow & E \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

Form the projective bundle, we have

$$\begin{array}{ccc} \mathbb{P}(L) & \longrightarrow & \mathbb{P}(E) \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

But $\mathbb{P}(L)$ is a bundle whose fibre is one point, hence it is the same as L, from which one can see that the projection from $\mathbb{P}(L)$ to Y is an isomorphism, and this gives a morphism from Y to $\mathbb{P}(E)$.
Another way to think of this is that, since $L \subseteq f^*(E)$, we map can the point in $y \in Y$ to the line $[L_y]$ in the corresponding projective space of the fibre $f(y)$, where L_y is the fibre over y.
In this case, the tautological bundle S pulls back to L, or rather isomorphic to L, since given a point $y \in Y$, the pull back of S is the line $f(y) \times [L_y] \times L_y$.
Or we may convince ourselves that a bundle morphism $\mathbb{P}(L) \to \mathbb{P}(E)$ pulls back the tautological bundle to tautological bundle, and the tautological bundle over $\mathbb{P}(L) \cong Y$ is L itself.

Come back to the last lecture, where we discussed morphisms to projective spaces. If we regard a vector space k^n as a fibre over a point, then $\mathbb{P}^n$ is nothing but the projective bundle. Thus a morphism $Y \to \mathbb{P}^n$ is equivalent to a line sub-bundle $L \subseteq Y \times k^n$. The dual of this line sub-bundle inherits n sections, namely the coordinate functions on $Y \times k^n$. The condition that L is a subbundle guarantees that these n sections do not vanish simultaneously.

We move to Grassmannians, which we have seen in Math 631. Let $0 < d < n$, $Gr(d,n) = \{\text{d-dimensional subspaces of } k^n\}$. This is an algebraic variety with open covers of the form $\to A^{d(n-d)}$. Its dimension is $d(n-d)$. (See Assignment 3 as well as Math 631 for more about this.)

There is also an interesting vector bundle on $Gr(d,n)$. The tautological sub-bundle $S \subseteq \mathbb{1}^n$ where we use $\mathbb{1}^n$ to denote the trivial bundle $Gr(d,n) \times k^n$. The situation is the same as before, the bundle is formed by $[W] \times W$ where W is a d-dimensional subspace defining a point in $Gr(d,n)$.

Before discussing the morphisms to Grassmanianns, we first recall the Plucker embedding:

$$Gr(d,n) \xrightarrow{\iota} \mathbb{P}(\wedge^d k^n)\ \ ,[W] \mapsto \wedge^d W$$

Then we have

$$\begin{array}{ccc} \wedge^d \mathbb{1}^n & = & \iota^* \mathbb{1}^m \\ \uparrow & & \uparrow \\ \wedge^d S & = & \iota^* \mathcal{O}(-1) \end{array}$$

where $m = \binom{n}{d}$ is the dimension of $\wedge^d k^n$. Thus $\wedge^d S$ is just the restriction of the tautological bundle of $\mathbb{P}(\wedge^d k^n)$.

Claim: $\mathrm{Hom}(Y, Gr(d,n)) = \{\text{subbundle of } \mathbb{1}_Y^n \text{ of rank d}\}$.

A brief discussion of the proof: Given a morphism $f : Y \to Gr(d,n)$, we pull back the tautological bundle to get the rank d sub-bundle.

Conversely, if we are given a rank d sub-bundle $F \subseteq \mathbb{1}_Y^n$, then $\wedge^d F \subseteq \wedge^d \mathbb{1}_Y^n$, which is a line sub-bundle of the trivial bundle whose rank is the same as $\mathbb{P}(\wedge^d k^n)$. The description of morphisms to projective spaces concludes that we have a morphism $Y \to \mathbb{P}(\wedge^d k^n)$. The map is given simply by $y \mapsto \wedge^d F_y$, where $y \in Y$ and F_y is the fibre over y. The image of this map actually lies in $Gr(d,n)$, since $\wedge^d F_y$ corresponds to the point F_y in $Gr(d,n)$.

In a word, the map is given by $y \to [F_y]$, which is not surprising, despite the lengthy description of it. The verification that the pull-back of the tautological bundle under the map we constructed is isomorphic to the original vector bundle is omitted.

We can use this description to revisit a well-known fact: $Gr(d,V) \cong Gr(n-d, V^*)$. V is a vector space of dimension n, and $Gr(d,V)$ is the variety consists of d-dimensional subspaces of V, V^* is the dual space of V.

The correspondence is given by $W \subseteq V$ to $(V/W)^* \subseteq V^*$.

Since $\mathrm{Hom}(Y, Gr(d,V)) = \{\text{vector sub-bundle of } Y \times V \text{ of rank d}\}$, and a vector sub-bundle $F \subseteq Y \times V$ corresponds to, after dualizing, a vector sub-bundle $(Y \times V^*)/F^* \subseteq Y \times V^*$, hence that set is in one-to-one correspondence with $\{\text{vector sub-bundle of } Y \times V^* \text{ of rank } n-d\}$. This establishes $\mathrm{Hom}(Y, Gr(d,V)) \cong \mathrm{Hom}(Y, Gr(n-d, V^*))$. By this universal property (an example of the Yoneda principle in category theory), then this gives a isomorphism between $Gr(d,V) \cong Gr(n-d, V^*)$.

Next, we extend the notion of Grassmannian to vector bundles.

Definition 52. Let $E \to X$ be a vector bundle of rank n, let $0 < d < n$, then the associated Grassmannian bundle $Gr(d,E) \to X$ is defined by gluing local coordinates. If $\{E|_{U_\alpha} \cong U_\alpha \times k^n\}$ is a trivialization of E, then $\{E|_{U_\alpha} \cong U_\alpha \times Gr(d,n)\}$ is a trivialization of $Gr(d,E)$. The transition map is induced from $U_{\alpha\beta} \times k^n \xrightarrow{\theta_{\beta\alpha}} U_{\beta\alpha} \times k^n$ to

$$U_{\alpha\beta} \times Gr(d,n) \xrightarrow{\bar{\theta}_{\beta\alpha}} U_{\beta\alpha} \times Gr(d,n) \ .$$

The induced map from an isomorphism of vector spaces $f : V \to W$ to Grassmanninas $Gr(d,V) \to Gr(d,W)$ is straight forward, taking any d-dimensional subspaces $F \subseteq V$ to $f(F) \subseteq W$.

As before, we

Claim: $\mathrm{Hom}(Y, Gr(d,E)) = \{f : Y \to X, \text{ together with a sub-bundle of } f^*(E) \text{ of rank d}\}$.

One can again use the fact that if F is a vector bundle of rank d on Y, then $Gr(d,F)=Y$ for half of this proof. In parallel, we also have a natural isomorphism of Grassmannian bundles: $Gr(d,E)\cong Gr(n-d,E^*)$, where n is the rank of the bundle E.

It is easy to, at least come across the generalization of Grassmannian bundles.

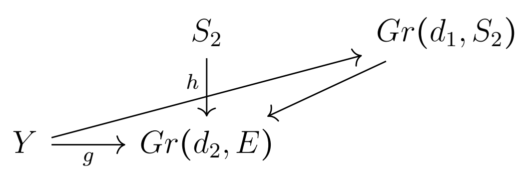
Definition 53. Given $0<d_1<d_2<...<d_r<n$, the flag bundle $Fl(\underline{d},E),\underline{d}=(d_1,d_2,...,d_r)$ associated to a vector bundle $p:E\to X$ is the variety satisfying the following universal property: $\mathrm{Hom}(Y,Fl(\underline{d},E))=\{f:Y\to X$ together with sub-bundles $S_1\subseteq S_2\subseteq...\subseteq S_r\subseteq f^*(E)$ of rank $\underline{d}=(d_1,d_2,...,d_r)\}$

If V is an n-dimensional vector space, view it as a vector bundle over a point, then $Fl(\underline{d},V)$ is called the flag variety. If we exhaust all dimensions of sub-spaces, then $Fl((1,2,...,n-1),V)$ is called the complete flag variety, sometimes realized as a "G/B".

The definition does not assure the existence of the flag variety. Let us construct it first. We will illustrate the process in the case $r=2$. The case $r=1$ is Grassmannian bundles.

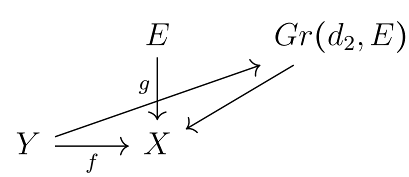
Let $r=2, 0<d_1<d_2<n$, we want to construct $Fl((d_1,d_2),E)$. Let S_2 be the tautological bundle of $Gr(d_2,E)$, define $Gr((d_2,d_2),E)=Gr(d_1,S_2)$. We will show that $Gr(d_1,S_2)$ satisfies the universal property. Intuitively, giving a morphism to $Gr(d_1,S_2)$ is equivalent to giving a morphism to $Gr(d_2,E)$ with a sub-bundle of S_2 of rank d_1. Now S_2 pulls back to a sub-bundle of $f^*(E)$ of rank d_2, and we get all the information.

Let $h:Y\to Gr(d_1,S_2)$ be a morphism, compose it with the projection we get $g:Y\to Gr(d_2,E)$ and $f:Y\to X$. First consider



The tautological bundle S_1 of $Gr(d_1,S_2)$ is a rank d_1 bundle and it pulls back to Y to be a sub-bundle $h^*(S_1)\subseteq g^*(S_2)$, by what we have done before about the characterization of morphisms to Grassmannians.

Next consider



The tautological bundle S_2 of $Gr(d_2,E)$ pulls back to a rank d_2 subbundle of $f^*(E)$. Now $h^*(S_1)\subset g^*(S_2)\subset f^*(E)$ gives the desired vector subbundles. The converse is to trace back these maps.

We can construct flag bundles using different orderings. For example, start with $Gr(d_1,E)$ and its tautological subbundle S_1 of $p^*(E)$. Then the Grassmann bundle $Gr(d_2-d_1,p^*(E)/S_1)$ has a tautological subbundle of the form S_2/S_1, which gives the required tautological bundles $S_1 \subset S_2 \subset E$. (Here, as is common, we have omitted notation for pullbacks!) The fact that they satisfy the same universal properties shows that the flag bundles constructed by using any orderings produce canonically isomorphic varieties.

Exercise Let $p : E \to X$ be a vector bundle, then we have the tautological bundle $\mathcal{O}(-1) \subseteq p^*(E)$ on $\mathbb{P}(E)$. Dualize it, we get $p^*(E^*) \to \mathcal{O}(1)$. Show that $\Gamma(\mathbb{P}(E), \mathcal{O}(d)) \cong \Gamma(X, Sym^d(E^*))$ where Γ is the global section, $Sym^d(E^*)$ is to take symmetric product of degree d on the fibre level. This is the analog of $\Gamma(\mathbb{P}^n, \mathcal{O}(d))$={Homogeneous polynomials of degree d}.

9. Tuesday, February 12th

Written by Jingchuan Xiao.

Definition 54. Let X be an algebraic variety (Cartan Space). An **ideal sheaf** is, for every open $U \subset X$, have an ideal $I(U) \subset O_X(U)$ satisfying
1) $U' \subset U$ open, if $f \in I(U)$, then $f|_{U'} \in I(U')$;
2) If $\{U_\alpha\}$ covers U, $f \in O_X(U)$ and $f|_{U_\alpha} \in I(U_\alpha)$ for all α, then $f \in I(U)$.

Example $Z \subset X$ closed. $I(U) = \{f \in \mathcal{O}_X(U) | f|_Z \equiv 0\}$. All $I(U) = \sqrt{I(U)}$.
Example X affine. $A = \mathcal{O}_X(X)$, any ideal $I \subset A$ gives ideal sheaf.

$$I(U) = \{f \in \mathcal{O}_X(U) |\ \textit{any } p \in U \textit{ has a nbhd } U' \textit{ where } f = \frac{g}{h},\ h \neq 0, g \in I, h \in A\}.$$

Lemma 55. *X affine. $I \to \mathcal{J}_X(X)$, $I_f \to \mathcal{J}_X(X_f)$ are isomorphisms (same proof as for $I = A$). Any ideal sheaf on X comes from a unique ideal I.*

We want to define "hypersurface". In $\mathbb{A}^n$, it corresponds to $f \in k[x_1, ..., x_n]$, $f \neq 0$. Say $f \in \mathcal{O}_X(U)$ is a non-zero-divisor if for all affine open $U' \subset U$, $f|_{U'}$ is a non-zero-divisor $\Leftrightarrow$ any such U', f does not vanish on any irreducible component of U'.

Definition 56. An **effective Cartier divisor** D on X is an ideal sheaf $\mathcal{J} \subset \mathcal{O}_X$ so for $U \subset X$ open, every point $p \in U$ has affine open neighborhood U' such that $\mathcal{J}_X(U') = (f)$, where f is an non-zero-divisor in $\mathcal{O}_X(U')$.

Example For $X = \prod^n$, any homogeneous polynomial $F \neq 0$ defines an effective Cartier divisor $V(F)$. On U_i, it is defined by $\frac{F}{x_i^d}$, $d = deg\ F$.

Any effective Cartier divisor is defined by data:

An (affine) open covering U_α of X with a non-zero-divisor $f_\alpha \in \mathcal{O}_X(U_\alpha)$ such that on $U_{\alpha\beta} = U_\alpha \bigcap U_\beta$, $(f_\beta) = (f_\alpha)$ in $\mathcal{O}_X(U_{\alpha\beta})$, $f_\beta = u_{\beta\alpha} f_\alpha$, $u_{\beta\alpha} \in \mathcal{O}_X(U_{\alpha\beta})^*$. This determines an effective Cartier divisor. When do 2 such data define the same effective Cartier divisor? Say when (U, f) is compatible with given data $(f|_{U \bigcap U_\alpha}) = (f_\alpha|_U)$.
Note: $u_{\alpha\alpha} = 1$. $u_{\gamma\alpha} = u_{\gamma\beta} u_{\beta\alpha}$ for α, β, γ, this is a transition function for a line bundle, denoted $O(D)$, where D is the Cartier divisor defined by data.
Exercise This is independent of choice of data.

In fact, we don't need to take local eqns f_α in $\mathcal{O}_X(U_\alpha)$; we can take them in $S^{-1}\mathcal{O}_X(U_\alpha)$, where S is the multiplicative set of non-zero-divisors in $\mathcal{O}_X(U_\alpha)$. Then

$(f_\alpha) \subset S^{-1}\mathcal{O}_X(U_\alpha)$ is a principal $\mathcal{O}_X(U_\alpha)$-module.

Exercise $\frac{f}{g}$ defines a Cartier divisor on $\to A^n$, where $f, g \in k[x_1, ..., x_n]$; $\frac{F}{G}$ defines a Cartier divisor on $\prod^n$, where $F, G \in k[x_1, ..., x_n]$ are homogeneous.

We still have data (U_α, f_α), $f_\beta = u_{\beta\alpha} f_\alpha$. It still defines a line bundle $O(D)$.

If effective Cartier divisors D, E in X correspond to ideals $\mathcal{J}_D$, $\mathcal{J}_E$, we have that the effective Cartier divisor $D+E$ correspond to $\mathcal{J}_D\mathcal{J}_E$. Similarly we have $D+E$ for arbitary Cartier divisor D, E. If D, E are with local data (U_α, f_α), (U_α, g_α), then $D+E$ has local data $(U_\alpha, f_\alpha g_\alpha)$. We get an abelian group $CDiv(X)$ of Cartier divisors on X ($D-E$ is defined by $(U_\alpha, f_\alpha/g_\alpha)$).

Exercise $O(D+E) = O(D) \otimes O(E)$.

We have an abelian group $Pic(X)$ of line bundles on X.

$$CDiv(X) \to Pic(X)$$

$$D \mapsto O(D)$$

Claim: On $\mathbb{P}^n$, $Pic(\mathbb{P}^n) = \mathbb{Z}$.

$$CDiv(\mathbb{P}^n) = \{Div(F/G) | F, G \text{ are homogeneous of degree } d, e, resp.\}$$

Let X be irreducible, $R(X)$ be the field of rational functions on X. Any $f \in R(X)^*$ defines a Cartier divisor $Div(f)$ with data (X, f) (or (U_α, f)).

If L is a line bundle on any X, s is a section of X that does not vanish identically on any irreducible component of any affine open $U \subset X$. Then $Z(s) \subset X$ is an effective Cartier Divisor.

Proposition 57. *X irreducible. The following sequence is exact:*

$$0 \longrightarrow \mathcal{O}_X(X)^* \longrightarrow R(X)^* \longrightarrow CDiv(X) \longrightarrow Pic(X) \longrightarrow 0$$

$$f \longmapsto Div(f)$$

$$D \longmapsto O(D)$$

Proof Mostly routine (**Exercise**). We only show the surjectivity: Given translation data U_α, $u_{\beta\alpha} \in \mathcal{O}_X(U_{\alpha\beta})$ for a line bundle L, choose α_0. Set $f_\alpha = u_{\alpha\alpha_0}$, $f_\beta = u_{\beta\alpha} f_\alpha$ ($f_\alpha \in R(X)^*$ by irreducibility). (U_α, f_α) defines a Cartier divisor D. $O(D) = L$.

Section of $O(D)$, D given by data (U_α, f_α), is given by a collection $s_\alpha = u_{\beta\alpha} s_\alpha$. Since $f_\beta = u_{\beta\alpha} f_\alpha$, $\frac{s_\alpha}{f_\alpha} = \frac{s_\beta}{f_\beta}$ on $U_\alpha \cap U_\beta$. If X is irreducible, this defines an $f \in R(X)^*$ where $f = \frac{s_\alpha}{f_\alpha}$ in U_α.

$$\Gamma(X, O(D)) = \{f \in R(X) | f_\alpha f \in \mathcal{O}_X(U_\alpha) \text{ for all } \alpha\}$$

I.e., f has poles controlled by D.

If D is effective, then $\Gamma(X, O(-D)) = \mathcal{J}_D(X)$.

Quotient Varieties (GIT)

① Let G be a finite group acting on any Cartan space X, say on left, we have a Cartan space X/G with morphism $\pi : X \to X/G$. X/G is the set of G-orbits given the quotient topology. If $U \subset X$ is G-invariant, G acts (on right) on $\mathcal{O}_X(U)$:

$$f^g(x) = f(gx).$$

For $U \subset X/G$ open, $\pi^{-1}(U)$ is open, invariant.

Set $O_{X/G}(U) = O_X(\pi^{-1}(U))^G = \{f \in O_X(\pi^{-1}(U)) | f^g = f, \forall g \in G\}$.

Proposition 58. *If X is an affine variety, then X/G is an affine variety with $X \to G$ corresponding to $A \hookleftarrow A^G$ (finite, integral).*

10. Thursday, February 14th

Written by Jack Carlisle.

Suppose that X is an irreducible variety. Then for any Cartier divisor D given by local data (f_α, U_α), we have a canonical isomorphism

$$\Gamma(X, O(D)) = \{f \in R(X) \mid f_\alpha f \in \mathcal{O}_X(U_\alpha) \text{ for all } \alpha\}$$

For $f \in R(X)$ as above, if we let $s_\alpha = f_\alpha f$, then s_α is a section of $O(D)$ over U_α. These s_α glue to give the corresponding section $s : X \to O(D)$. If we write

$$\text{Zero}(s) = \{x \in X \mid s(x) = 0\}$$

Then $\text{Zero}(s)$ is obtained by gluing together the $\text{Zero}(s_\alpha) \subset U_\alpha$.

Example: If D is effective, then $f = 1$ is in $\Gamma(X, O(D))$, and we have

$$\text{Zero}(1) = \cup_\alpha \{f_\alpha = 0\} = D$$

If X is irreducible and $D = (U_\alpha, f_\alpha)$ is a Cartier divisor on X, then D determines a **Weil divisor** $[D] = \sum m_i V_i$. This is a finite $\mathbb{Z}$-linear combination of codimension 1 irreducible subvarieties of X. In fact, this determines a homomorphism

$$\begin{aligned} CDiv(X) &\to WDiv(X) \\ D &\mapsto [D] \\ D \pm E &\mapsto [D \pm E] = [D] \pm [E] \end{aligned}$$

Lets discuss how this Weil divisor $\sum m_i V_i$ is defined. For any 1-dimensional Noetherian domain A, any nonzero $f \in A$ has an order $\text{ord}(f) = \ell(A/(f))$, the length of the ring $A/(f)$. This satisfies $\text{ord}(fg) = \text{ord}(f) + \text{ord}(g)$. (Can you write down an exact sequence so lengths add to show this?) Because of this one can extend the order function to a homomorphism ord: $\mathrm{Frac}(A)^* \to \mathbb{Z}$, with $\mathrm{Frac}(A)$ the quotient field of A, by $\mathrm{ord}(f/g) = \mathrm{ord}(f) - \mathrm{ord}(g)$. (Check that this is a well- defined homomorphism.)

For a general Cartier divisor D on an irreducible X, and any irreducible closed subvariety V of X of codimension one, define $\mathrm{ord}_V(D)$ to be $\mathrm{ord}(f)$, where f defines D in any affine open U meeting V, using A the local ring of V on X, which is the localization of $O_X(U)$ at the prime ideal corresponding to $V \cap U$.

The Weil divisor $[D]$ of D is $\sum m_V V$, the sum over all irreducible closed subvarieties V of X for which the order $m_V = \mathrm{ord}_V(D)$ is not zero. Without any finiteness assumptions on X, this could be an infinite formal sum, but at most finitely many V meet any fixed affine open set.

Fact: If X is locally factorial (for example: non-singular), then $CDiv(X) = WDiv(X)$.

Moving on, we return to group actions on varieties.

Lemma: If k is a Noetherian ring, A a finitely generated k-algebra, G a finite group acting on A, then $A^G = \{f \in A \mid f^g = f \text{ for all } g \in G\}$ is a finitely generated k-algebra.

Proof: For any $f \in A$, we have the equation $\prod_{g \in G}(T - f^g)$ in $A^G[T]$. Take a finite set of generators of A as a k-algebra, and take all of the coefficients of their equations. These generate a (finitely generated) k-algebra $B \subset A^G \subset A$. We know that B is Noetherian since it is finitely generated over the Noetherian ring A, and we also know that A is a finite B-module. This implies that A^G is a finite B-module, from which we deduce that A^G is finitely generated over k as a k-algebra. □

Lets apply this theorem. Suppose that $X = \mathrm{Specm}(A)$ is an affine variety, and G is a finite group acting on the left. Then we can form the Cartan orbit space X/G, and this comes with a map of Cartan space $X \to X/G$.

Claim: The There is a canonical identification $X/G = \mathrm{Specm}(A^G)$, and the following square commutes

$$\begin{array}{ccc} \mathrm{Specm}(A) & \longrightarrow & \mathrm{Specm}(A^G) \\ \| & & \| \\ X & \longrightarrow & X/G \end{array}$$

where $\mathrm{Specm}(A) \to \mathrm{Specm}(A^G)$ corresonds to $A^G \subset A$.

Proof: If x and x' are in different G-orbits, then since $x \neq x'$, there exists $f \in A$ such that $f(x) = 0$ and $f(x') \neq 0$. Then the element $f^G := \prod_{g \in G} f^g \in A^G$ is such that $f^G([x]) = 0$ and $f^G([x']) \neq 0$. The rest of the proof is left as an exercise. □

Example: Let $X = \mathbb{A}^2$, $G = \mathbb{Z}/2\mathbb{Z}$, and let G act on X by $v \mapsto -v$. If $\operatorname{char} k \neq 2$, then the orbit of any nonzero point in $\mathbb{A}^2$ has size 2, and the orbit of $0 \in \mathbb{A}^2$ is a singleton. We have $k[X,Y]^G = k[X^2, XY, Y^2] \cong k[u,v,w]/(v^2 - uw)$. Note that if $\operatorname{char} k = 2$, then the action of G is trivial, so the map corresponding to $k[X^2, XY, Y^2] \to k[X,Y]$ is bijective but not an isomorphism.

Exercise: Suppose that G is a finite group acting on a separated variety X such that every orbit of G on X is contained in some (invariant) affine open $U \subset X$. Then the quotient $X \to X/G$ is an algebraic variety. (We use the separated hypothesis to start with any such open affine U and replace it with $\cap_{g \in G} g(U)$, which is affine since X is separated.)
Example: Any quasi-projective variety X works.

Note that we can form the quotient Cartan space X/G when G is any group, not necessarily a finite group.

Example: Let $X = \mathbb{A}^{n+1} = \{0\}$ with $G = k^*$. Then G acts on X by $t \cdot (x_0, \ldots, x_n) = (tx_0, \ldots, tx_n)$, and we have

$$(\mathbb{A}^{n+1} - \{0\})/k^* = \mathbb{P}^n$$

Example: Let $X = \to A^{n+1}$ and let $G = k^*$ with the action $t(x_0, \ldots, x_n) = (tx_0, \ldots, tx_n)$. Then

$$\mathbb{A}^{n+1}/k^* = \mathbb{P}^n \cup \{*\}$$

This is a terrible space! The point $*$ is in the closure of any point $(x_0 : \cdots : x_n) \in \mathbb{P}^n$.

Example: Let $G = k^*$ act on $\mathbb{A}^2$ by $t \cdot (x,y) = (tx, t^{-1}y)$. Then $A = k[X,Y]$, and $A^G = k[XY] = k[T]$. The map $A^G \subset A$ corresponds to the map

$$\begin{aligned} \mathbb{A}^2 &\to \mathbb{A}^1 \\ (x,y) &\mapsto xy \end{aligned}$$

The fiber over $c \in \mathbb{A}^1$ is $\{xy = c\} \subset \mathbb{A}^2$. For $c \neq 0$, the fiber $\{xy = c\}$ is an orbit, but for $c = 0$, the fiber $\{xy = 0\}$ is the union of 3 orbits (which ones?).

Example: For G finite, $\pi : X \to X/G$ satisfies the universal property of the quotient: For any $\varphi : X \to Z$ mapping G-orbits to points, there exists a unique map $\psi : X/G \to Z$ so that

$$\begin{array}{ccc} X & \xrightarrow{\pi} & X/G \\ & \overset{\varphi}{\searrow} & \Big\downarrow{\scriptstyle\psi} \\ & & Z \end{array}$$

For the categorically minded: $\pi : X \to X/G$ is the colimit of the diagram

$$X \underset{g_1 \cdot}{\overset{g_n \cdot}{\underset{\vdots\; g_2 \cdot}{\rightrightarrows}}} X \xrightarrow{\pi} X/G$$

Where $G = \{g_1, \ldots, g_n\}$.

Next, we'll define wheighted projective space:

Definition: (Wheighted Projective Space) Let $q_\bullet = (q_0, \ldots, q_n)$ be a sequence of positive integers. We define a variety $\mathbb{P}(q_\bullet)$ by

$$\mathbb{P}(q_\bullet) = \mathbb{P}(q_0, \ldots, q_n) = \mathbb{A}^{n+1} - \{0\}/k^*$$

Where $t \cdot (x_0, \ldots, x_n) = (t^{q_0} x_0, \ldots, t^{q_n} x_n)$.

Claim: $\mathbb{A}^{n+1} - \{0\} \to \mathbb{P}(q_\bullet)$ is algebraic.

Proof: Exercise.

In fact if we define$k(q_\bullet) := k[X_0, \ldots, X_n]$ with $\dim X_i = q_i$, then $\mathbb{P}(q_\bullet) = \mathrm{Projm}(k(q_\bullet))$. In fact, we can realize $\mathbb{P}(q_\bullet)$ as a toric variety. Take $M = \mathbb{Z}^{n+1}$ with basis $e_0, \ldots, e_n$. Let Δ = the convex hull of $\frac{e_0}{q_0}, \ldots, \frac{e_n}{q_n}$. The algebra $S = S_\delta$ is exactly $k(q_\bullet)$, so $\mathbb{P}(q_\bullet) = X_\Delta$.

Next on the agenda is to construct $\mathbb{P}(q_\bullet)$ as an algebraic variety. We have the following commutative diagram of sets:

$$\begin{array}{ccc} \mathbb{A}^n = \{(x_0, \ldots, 1, \ldots, x_n)\} & \hookrightarrow & \{x_i \neq 0\} \subset \mathbb{A}^{n+1} - \{0\} \\ \Big\downarrow & & \Big\downarrow \\ \mathbb{A}^n/\mu_{q_i} & \xrightarrow{\cong} & \{x_i \neq 0\} \subset \mathbb{P}(q_\bullet) \end{array}$$

Where $\mu_{q_i} = \{t \in k^* \mid t_i^q = 1\}$ is the group of q_ith roots of unity. The corresponding maps of rings are

$$\begin{array}{ccc} k[x_0,\ldots,\widehat{x_i},\ldots,x_n] & \longleftarrow & S_{x_i} \\ \uparrow & & \uparrow \\ k[x_0,\ldots,\widehat{x_i},\ldots,x_n]^{\mu_{q_i}} & \xleftarrow[\cong]{} & S_{(x_i)} \end{array}$$

11. Tuesday, February 19th

12. Tuesday, February 19th

Before we start, there is a correction to be made in the definition of an effective Cartier divisor. D is an effective Cartier Divisor on X if there is an open covering $X = \bigcup_\alpha U_\alpha$ such that $D\big|_\alpha =< f_\alpha >$. That is, on any affine open set $U \subset X$, the ideal sheaf is only *locally* principal and not necessarily principal. (Just like how a vector bundle has an affine covering where it is trivial, but on an arbitrary affine open subset, it is only locally trivial).

We continue from where we left off on Thursday.

12.1. **Weighted Projective Space (cont.)** For $x = (x_0,\ldots,x_n) \in \mathbb{A}^{n+1} \setminus \{0\}$, the cones, prime ideal $\mathfrak{p}_x$ in S is the kernel of $S \to K[T]$ via $x_i \mapsto x_i T^{q_i}$. So $S/\mathfrak{p}_x$ is the subalgebra of $k[T]$ generated by T^{q_i} such that $x_i \neq 0$.

What are the generators of $\mathfrak{p}_x$? Note that as varieties, $\mathbb{P}(q_0,\ldots,q_n) \cong \mathbb{P}(dq_1,\ldots dq_n)$ via $\text{Projm}(S) \cong \text{Projm}(S^{(d)})$. If $d = gcd(q_0,\ldots,\hat{q}_i,\ldots,q_n)$ is relatively prime to q_i, then $\mathbb{P}(q_0,\ldots,q_n) \cong \mathbb{P}(\frac{q_0}{d},\ldots,q_i,\ldots,\frac{q_n}{d})$ because $S^{(d)} = k[x_0,\ldots,x_i^d,\ldots,x_n]$. So we can assume that the greatest common divisor of any n of the q_i's are 1.

Exercise. $\mathbb{P}(q.) = \mathbb{P}^n/\mu_{q_0} \times \ldots \mu_{q_n}$

Subvarieties of $\mathbb{P}(q.)$ are given by homogeneous (radical) ideals $I \subset K(q.) = S$ (where $I \not\subset S_+$). For Y an affine variety, $A = \mathcal{O}_Y(Y)$, subvarieties of $Y \times \mathbb{P}. \leftrightarrow$ homogeneous ideals $I \subset A(q.) = A[x_0,\ldots,x_n]$ where $\deg(x_i) = q_i$.

This construction gives us all the Projm's we have constructed so far, including all toric varieties X_Δ.

We have assumed that the $char(k) \nmid q_i$. This is because for any positive integer q, μ_q, the q^th roots of unity $=$ $\text{Specm}(\frac{k[T]}{\langle T^q - 1\rangle})$ is not reduced if $char(k) \mid q$. However, if one is careful, what we have done so far works for all fields.

12.2. **Toric Varieties (Part two).** We are given a convex polytope Δ in $M_\mathbb{R}$. We shall produce an affine covering for X_Δ. Assume Δ has vertices in M. Then, each vertex u will correspond to $\text{Specm}(S_{(\chi^u)})$.

Let $N = \text{Hom}(M,\mathbb{Z})$, the dual lattice of M and $\langle,\rangle : M \times N \to \mathbb{Z}$ be the natural pairing. For each cone $\sigma \subset M_\mathbb{R}$, we have a dual cone $\sigma^\vee = \{v \in N_\mathbb{R} | \langle u,v\rangle \geq 0, \forall u \in \sigma\}$.

We list some facts about polyhedra:

<u>Fact 1:</u> If σ is a rational polyhedral cone with vertices in M, then $\sigma^\vee$ is a rational polyhedral cone in N. Furthermore, $(\sigma^\vee)^\vee = \sigma$.

Definition 59. A face of a rational polyhedral cone σ in N is a subset of the form $u^\perp \cap \sigma$, for some $u \in \sigma^\vee \cap M$.

Fact 2: The face is also a rational polyhedral cone, generated by those generators v_i of σ such that $\langle u, v \rangle = 0$. The face of a face is a face. Intersection of 2 faces is a face. Any proper face is contained in a facet (a codimension 1 face).

Fact 3: If $\tau = u^\perp \cap \sigma$ is a face, then $\tau^\vee = \sigma^\vee + \mathbb{R}_{\geq 0}(-u)$.

Fact 4: Let σ and σ' be rational polyhedral cones in N. Assume that $\tau = \sigma \cap \sigma'$ is a face of each. Then there exists $u \in M$ with $u \in \sigma^\vee$ and $-u \in (\sigma')^\vee$ such that $\tau = u^\perp \cap \sigma$ and $\tau = (-u)^\perp \cap \sigma'$.

Definition 60. For σ, a rational polyhedral cone in N, we have an affine variety U_σ where $U_\sigma = \mathrm{Specm}(K[\sigma^\vee \cap M])$.

If $\tau = u^\perp \cap \sigma$ is a face of σ, then $U_\tau = \mathrm{Specm}(K[\tau^\vee \cap M]) = \mathrm{Specm}(K[\sigma^\vee \cap M]_{\chi^u})$, where the last equality is due to Fact 3. So $U_\tau \subset U_\sigma$ is a principal affine open subset. If $\tau = \sigma \cap \sigma'$ (as in Fact 4), then U_τ is a principal affine open subset of both U_σ and $U_{\sigma'}$ (with χ^u inverted in both).

Definition 61. A fan Σ in N is a collection of rational polyhedral cones such that

(1) A face of $\sigma \in \Sigma$ is also in Σ.
(2) If σ and σ' are in Σ, then $\sigma \cap \sigma'$ is a face of each (thus in Σ).

Exercise. *For any fan Σ, the affine varieties glue together to form a variety $X(\Sigma)$.*

We have a torus $T_N = \mathrm{Specm}(K[M]) = U_{\{0\}}$, an algebraic group via $K[M] \to K[M] \otimes K[M]$, where $\chi^u \mapsto \chi^u \otimes \chi^u$. Note that $T_N \cong (K^*)^n$ and it acts on all U_σ: $K[U_\sigma \cap M] \to K[M] \otimes K[\sigma^\vee \cap M]$, with $\chi^u \mapsto \chi^u \otimes \chi^u$. These glue together to give an action of T_N on $X(\Sigma)$, a "toric variety".

13. Thursday, February 21st

Written by Michael Mueller.

Using the notation from last time, let M be a lattice and let $N = \mathrm{Hom}(M, \mathbb{Z})$ be its dual lattice. Consider a fan Σ in N. For each rational polyhedral cone $\sigma \in \Sigma$, we have an affine variety

$$U_\sigma = \mathrm{Specm}(k[\sigma^\vee \cap M]).$$

These affine varieties U_σ (for all σ a rational polyhedral cone in Σ) glue together to form a toric variety $X(\Sigma)$.

Example 62. Let Δ be a convex polytope in $M_\mathbb{R}$ whose vertices lie in $M_\mathbb{Q}$, and assume that 0 is contained in the interior of Δ. Define

$$\Delta^* := \{v \in N_\mathbb{R} : \langle u, v \rangle \geq -1 \text{ for all } u \in \Delta\}.$$

Then Δ^* is a convex polytope in $N_\mathbb{R}$ and $(\Delta^*)^* = \Delta$. Letting Σ be the fan consisting of cones over faces of Δ^*, we have

$$X_\Delta = X(\Sigma).$$

Suppose we have a homomorphism of lattices (i.e., of abelian groups) $\varphi : N' \to N$. Dualizing this map, we get a corresponding homomorphism $\varphi^* : M \to M'$ where $M = \text{Hom}(N, \mathbb{Z})$ and $M' = \text{Hom}(N', \mathbb{Z})$. This then gives us a map of k-algebras $k[M] \to k[M']$, which finally corresponds to a map

$$\varphi_* : T_{N'} = \text{Specm}(k[M']) \to T_N = \text{Specm}(k[M])$$

by applying the contravariant functor Specm. This map $T_{N'} \to T_N$ is not just a map of varieties, but is in fact a map of **algebraic groups** (where $T_{N'}$ and T_N have the algebraic group structure discussed last class).

Example 63. Letting N be a lattice, a homomorphism $\varphi : \mathbb{Z} \to N$ corresponds to a homomorphism $T_{N'} \to T_N$, where

$$T_{N'} = T_{\mathbb{Z}} = \text{Specm}(k[\text{Hom}(\mathbb{Z}, \mathbb{Z})]) \cong \text{Specm}(k[\mathbb{Z}]) \cong \text{Specm}(k[T, T^{-1}]) = \mathbb{A}^1 - \{0\}.$$

If we are working over $\mathbb{C}$, then this means that homomorphisms $\mathbb{Z} \to N$ correspond to **one-parameter subgroups**, i.e. maps $\mathbb{C}^* \to T_N$. Similarly, homomorphisms $N \to \mathbb{Z}$ correspond to **characters**, or maps $T_N \to \mathbb{C}^*$.

Suppose we have fans Σ in N and Σ' in N' with a homomorphism $N' \xrightarrow{\varphi} N$. Assume the condition:

$(*)$ For each cone σ' in N' there is some cone σ in N such that $\varphi_{\mathbb{R}}(\sigma') \subset \sigma$.

(Here $\varphi_{\mathbb{R}} : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ is given by $\varphi_{\mathbb{R}} = \varphi \otimes \mathbb{R}$.) Then we get a morphism $\varphi_* : X(\Sigma') \to X(\Sigma)$ sending $U_{\sigma'}$ to U_σ by

$$k[\sigma^\vee \cap M] \to k[\sigma'^\vee \cap M'].$$

Moreover this morphism is equivariant, i.e. the following diagram commutes:

$$\begin{array}{ccc} T_{N'} \times X(\Sigma') & \longrightarrow & X(\Sigma') \\ \downarrow{\scriptstyle \varphi_* \times \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ T_N \times X(\Sigma) & \longrightarrow & X(\Sigma) \end{array}$$

Equivalently, $\varphi_*(t' \cdot x') = \varphi_*(t')\varphi_*(x')$ for $t' \in T_{N'}$ and $x' \in X(\Sigma')$.

Fact: Let $N' = N$, and let Σ' be a refinement of Σ. Then we have a morphism $\varphi_* : X(\Sigma') \to X(\Sigma)$ corresponding to the identity $\varphi : N \to N$, and we can use this to resolve singularities of all toric varieties (of all characteristics).

Note: Every lattice N has, for every integer n, a homomorphism

$$\varphi_n : N \to N, \quad \varphi_n(v) = nv.$$

If σ is a cone in N, then $(\varphi_n)_{\mathbb{R}}(\sigma) = \sigma$, so we always get a map $(\varphi_n)_* : X(\Sigma) \to X(\Sigma)$ for every fan Σ. We can view this as a generalization of the Frobenius map in characteristic p, as in fact when $n = p$ and the field k has characteristic p, $(\varphi_p)_*$ turns out to be the Frobenius map. (For a reference, see Burt Totaro's paper "Chow Groups, Chow Cohomology, and Linear Varieties".)

Example 64. Any rational polyhedral cone σ has a smallest face γ. For a fan Σ, all cones have the same smallest face γ, which is a subspace. In fact, $N' = \gamma \cap N$ is a lattice, with $\gamma = N'_{\mathbb{R}}$.

The torus $T_{N'}$ acts trivially on $X(\Sigma)$, so we get an action of $T_{\overline{N}} = T_N/T_{N'}$ where $\overline{N} = N/N'$. All of the $\sigma^\vee \cap M$ (for $\sigma \in \Sigma$) are in $\overline{M} \subset M$, where

$$\overline{M} = (\overline{N})^* \subset N^* = M.$$

The images $\overline{\sigma}$ of each σ in $\overline{N}$ form a fan $\overline{\Sigma}$, and in fact

$$X(\Sigma) = X(\overline{\Sigma}).$$

Note that all the cones $\overline{\sigma}$ are pointed, i.e. the line $\{0\}$ is a face. In the pointed case, $T_N = U_{\{0\}} \subset X(\Sigma)$ and so T_N has an orbit $U_{\{0\}}$ isomorphic to T_N (this is known as the "torus embedding").

Fact: The toric variety $X(\Sigma)$ is normal.

Fact: Any normal variety X of finite type which has a torus action with a dense open orbit comes from a fan, i.e.

$$X = X(\Sigma)$$

for some fan Σ. This is philosophically interesting, but perhaps not very useful.

For any rational convex polyhedron Δ in M, we constructed the toric variety X_Δ. In fact, for each face Γ of Δ we have a cone

$$\sigma_\Gamma = \{\sigma \in N_{\mathbb{R}} : \langle u, \sigma\rangle \le \langle u', v\rangle \text{ for all } u \in \Gamma,\ u' \in \Delta\}.$$

Let Σ consist of the cones over "faces" of this. It is then a fact that $X(\Sigma)$ is a proper variety, but not projective ($X(\Sigma)$ can be nonsingular).

Example 65. Let $N = \mathbb{Z}^2$ and let $\sigma_n = \langle (n,1), (n+1,1), \dots\rangle$. If we form the fan Σ, then $X(\Sigma)$ is irreducible and nonsingular, but it is not of finite type.

Exercise. *Assume σ is a pointed rational polyhedral cone. Then U_σ is nonsingular if and only if $U_\sigma \cong \mathbb{A}^r \times (k^*)^{n-r}$, if and only if σ is spanned by part of a basis of N. (For a reference, see Professor Mustata's notes.)*

13.1. **Presheaves and Sheaves.**

Definition 66. A presheaf $\mathcal{F}$ (of sets, groups, abelian groups, or anything else) on a space X is an assignment of sets (or groups, abelian groups, ...) $\mathcal{F}(U)$ to each open $U \subset X$ together with, for each $U' \subset U \subset X$ open, a morphism

$$\mathcal{F}(U) \xrightarrow{\rho_{U'U}} \mathcal{F}(U')$$

such that ρ_{UU} is the identity and

$$U'' \subset U' \subset U \Rightarrow \rho_{U''U} = \rho_{U''U'} \circ \rho_{U'U}.$$

Warning: There is a common abuse of notation, in which $\rho_{U'U}(f)$ is written instead as $f|_{U'}$ and is called the "restriction of f to U'". This is not actually a restriction (in general), so be aware that treating $\rho_{U'U}(f)$ as a restriction of a function (when f is not necessarily even a function, for example) is a potential point of confusion.

Example 67. Let A be a constant set (or group, abelian group, ...) and define $\mathcal{F}(U) = A$ for all U, with $\rho_{U'U} = \mathrm{Id}_A$ for all $U' \subset U$. Then this is a presheaf, known as a **constant presheaf**.

Example 68. Let $(X, \mathcal{O}_X)$ be a Cartan space. Then $\mathcal{O}_X$ is a presheaf.

Example 69. Let X be an algebraic variety, let E be a vector bundle on X, and for an open set U define

$$\Gamma(U, E) = \{\text{sections of } E \text{ on } U\}.$$

Then $U \mapsto \Gamma(U, E)$ is a presheaf (with the obvious restriction morphisms).

Example 70. Let $\pi : Y \to X$ be a continuous map of spaces. Then

$$\mathcal{F}(U) = \{s : U \to Y \text{ where } \pi \circ s = \mathrm{Id}_U\},$$

together with the obvious restrictions, is a presheaf of sets.

Example 71. Let X be a variety and let $\mathcal{J} \subset \mathcal{O}_X$ be an ideal sheaf. Then $\mathcal{J}$ is a presheaf.

Example 72. With $\mathcal{J} \subset \mathcal{O}_X$ an ideal sheaf, $U \mapsto \mathcal{O}_X(U)/\mathcal{J}(U)$ (together with canonical restrictions) forms a presheaf.

Definition 73. Let $\mathcal{F}$ be a presheaf. Then $\mathcal{F}$ is a **sheaf** if for any open cover $U = \bigcup_\alpha U_\alpha$, given $f_\alpha \in \mathcal{F}(U_\alpha)$ for all α such that

$$\rho_{U_{\alpha\beta} U_\alpha}(f_\alpha) = \rho_{U_{\alpha\beta} U_\beta}(f_\beta),$$

there exists a unique $f \in \mathcal{F}(U)$ such that $\rho_{U_\alpha U}(f) = f_\alpha$.

Exercise. *All of the examples of presheaves thus far are sheaves with the exception of the first example (a constant presheaf) and the last example ($U \mapsto \mathcal{O}_X(U)/\mathcal{J}(U)$).*

14. Tuesday, February 26th

Written by Bradley Zykoski.

Today we discuss several examples of (pre)sheaves and further constructions that one can make with (pre)sheaves.

Example 74. Let X be any topological space, let A and B be abelian groups, and let $\phi : A \to B$ be a homomorphism that is not an isomorphism. Set $\mathcal{F}(U) = A$ if $U = X$, and set $\mathcal{F}(U) = B$ if $U \neq X$. Let $\rho_{U'U} = \phi$ for $U' \neq X$ and $U = X$, and $\rho_{U'U} = \mathrm{Id}$ otherwise. $\mathcal{F}$ is a presheaf, but not necessarily a sheaf. Taking an open cover $\{U_i\}_{i \in I}$ of X with $U_i \neq X$, we have $\mathcal{F}(U_i) = B$, and $\mathcal{F}(U_i \cap U_j) = B$. A is not necessarily the kernel of the map $\prod_i \mathcal{F}(U_i) \to \prod_{i,j} \mathcal{F}(U_i \cap U_j)$ sending $(b_i)_{i \in I}$, $b \in B$ to $(b_i - b_j)_{i,j}$, which does not involve A, which would be necessary for $\mathcal{F}$ to be a sheaf.

Example 75. Let $\pi : Y \to X$ be a map of topological spaces, and for each open subset U of X, let $\mathcal{F}(U)$ be the set of sections of π over U. The restriction maps for $\mathcal{F}$ are clear. Then $\mathcal{F}$ is a sheaf of sets. If Y has additional structure, then so may the sheaf $\mathcal{F}$. For example, if we have $Y \times_X Y \xrightarrow{\cdot} Y$ and $Y \xrightarrow{-1} Y$ making the fibers of Y into abelian groups, then $\mathcal{F}$ is a sheaf of abelian groups.

Definition 76. Let $(X, \mathcal{O}_X)$ be a topological space X equipped with a (pre)sheaf $\mathcal{O}_X$ of rings. We say that a (pre)sheaf $\mathcal{F}$ of abelian groups is a *(pre)sheaf of $\mathcal{O}_X$-modules* (or simply an *$\mathcal{O}_X$-module*) if for every open U in X we have a map $\mathcal{O}_X(U) \times \mathcal{F}(U) \xrightarrow{\cdot} \mathcal{F}(U)$ making $\mathcal{F}(U)$ an $\mathcal{O}_X(U)$-module such that for $U' \subseteq U$, the diagram

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{O}_X(U') \times \mathcal{F}(U') & \longrightarrow & \mathcal{F}(U') \end{array}$$

commutes.

Example 77. If X is an algebraic variety and E is a vector bundle on X, then $U \mapsto \Gamma(U, E)$ is a sheaf of $\mathcal{O}_X$-modules. Indeed, it is a locally free $\mathcal{O}_X$-module.

Example 78. If $\mathcal{I} \subseteq \mathcal{O}_X$ is an ideal sheaf, then $\mathcal{I}$ is a sheaf of $\mathcal{O}_X$-modules.

Example 79. For $f : Y \to X$ a morphism of Cartan spaces, $U \mapsto \mathcal{O}_Y(f^{-1}(U))$ is a sheaf of $\mathcal{O}_X$-algebras on X. The restriction maps for this sheaf are clear. Indeed, for any map $f : Y \to X$ of topological spaces and any (pre)sheaf $\mathcal{F}$ on Y, we have a (pre)sheaf $U \mapsto \mathcal{F}(f^{-1}(U))$ on X. This is called the *direct image* of $\mathcal{F}$, and is written $f_*(\mathcal{F})$.

Exercise. *If $\mathcal{F}$ is a sheaf, then so is $f_*(\mathcal{F})$.*

Proof. Take an open cover $\{U_i\}_{i\in I}$ of an open set U in X. Take $f_i \in f_*(\mathcal{F})(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i\cap U_j}$ for each i, j. Now $f_i \in \mathcal{F}(f^{-1}(U_i))$ and $f_i|_{f^{-1}(U_i)\cap f^{-1}(U_j)} = f_j|_{f^{-1}(U_i)\cap f^{-1}(U_j)}$ for each i, j. $\{f^{-1}(U_i)\}_{i\in I}$ is an open cover of $f^{-1}(U)$, since for each $y \in f^{-1}(U)$, $f(y) \in U$, so that $f(y) \in U_i$ for some $i \in I$. Since Y is a sheaf, there is a unique $f \in \mathcal{F}(f^{-1}(U))$ such that $f|_{f^{-1}(U_i)} = f_i$. Then $f \in f_*(\mathcal{F})(U)$ and $f|_{U_i} = f_i$, as desired. □

Exercise. *Suppose we have a functorial assignment $V \mapsto \mathcal{F}(V)$ for sets V in a basis $\mathcal{B}$ of X that satisfies the analogues of the sheaf axioms. If for any arbitrary open U in X, we set*

$$\mathcal{F}(U) = \varprojlim_{\substack{V \subseteq U \\ V \in \mathcal{B}}} \mathcal{F}(V),$$

show that $\mathcal{F}$ is a sheaf.

Definition 80. A (pre)sheaf $\mathcal{G}$ is a *sub(pre)sheaf* of $\mathcal{F}$ if for $U' \subseteq U$ open in X, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{G}(U) & \hookrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{G}(U') & \hookrightarrow & \mathcal{F}(U'). \end{array}$$

Example 81. Let $(X, \mathcal{O}_X)$ be a Cartan space. Then the constant presheaf k and any ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$ are sub(pre)sheaves of $\mathcal{O}_X$.

Definition 82. A *morphism* $\phi : \mathcal{F} \to \mathcal{G}$ of (pre)sheaves is a natural transformation of functors. That is to say, it assigns to each open U in X a homomorphism $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ such that for every $U' \subseteq U$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(U') & \xrightarrow{\phi(U')} & \mathcal{G}(U'). \end{array}$$

Note that if $\mathcal{F}$ and $\mathcal{G}$ are sheaves of abelian groups, and $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then $\ker(\phi)$ is a subsheaf of $\mathcal{F}$, where $\ker(\phi)(U) \coloneqq \ker(\phi(U))$. We also have presheaves

$$\text{“Im}(\phi)\text{”}(U) = \operatorname{Im}(\phi(U)), \quad \text{“coker}(\phi)\text{”}(U) = \operatorname{coker}(\phi(U)),$$

but these are not sheaves in general, and so a better construction of images and cokernels is desired.

Exercise. *Show that “Im” and “coker” need not be sheaves, even if $\mathcal{F}$ and $\mathcal{G}$ are. In particular, if $\mathcal{I} \subseteq \mathcal{O}_X$ is an ideal sheaf, then the “cokernel” $U \mapsto \mathcal{O}_X(U)/\mathcal{I}(U)$ of the inclusion $\mathcal{I} \hookrightarrow \mathcal{O}_X$ is not a sheaf.*

Definition 83. Let X be an algebraic variety. Define

$$R_X(U) = \varinjlim_{\substack{U' \subseteq U \\ U' \text{ dense in } U}} \mathcal{O}_X(U').$$

Exercise. *This is a sheaf.*

Exercise. *If X is irreducible, then R_X is the constant sheaf $R(X)$ = the field of rational functions on X.*

Definition 84. If $\mathcal{F}$ is any presheaf on a topological space X, then for any $x \in X$, we have the *stalk* of $\mathcal{F}$ at x, given by

$$\begin{aligned} \mathcal{F}_x &\coloneqq \{\text{germs at } x\} \\ &= \{f \in \mathcal{F}(U) \text{ for } U \ni x\}/(f \in \mathcal{F}(U) \sim g \in \mathcal{F}(V) \iff \exists U' \subseteq U \cap V \text{ with } \rho_{U'U}(f) = \rho_{U'V}(g)) \\ &= \varinjlim_{U \ni x} \mathcal{F}(U). \end{aligned}$$

Any $f \in \mathcal{F}(U)$ for $U \ni x$ determines $f_x \in \mathcal{F}_x$, and so f gives a “function” on U assigning $x \mapsto f_x \in \mathcal{F}_x$.

Exercise. *If $\mathcal{F}$ is a sheaf, then this function determines f.*

Exercise. *(Important!) If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of abelian groups, and the induced map $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism, then ϕ is an isomorphism of sheaves.*

Definition 85. Given a presheaf on X, define a presheaf $\mathcal{F}^\#$ on X by

$$\begin{aligned} \mathcal{F}^\#(U) &\coloneqq \left\{ s : U \to \coprod_{x \in X} \mathcal{F}_x \;\middle|\; s(x) \in \mathcal{F}_x \; \forall x \in U \right\} \\ &= \prod_{x \in U} \mathcal{F}_x. \end{aligned}$$

This is obviously a presheaf, with the obvious restriction maps.

Exercise. *The presheaf $\mathcal{F}^{\#}$ is always a sheaf.*

Given a presheaf $\mathcal{F}$ on X, we will construct the *sheafification of* $\mathcal{F}$ / the *associated sheaf* $\mathcal{F}^{+}$, with a morphism $\mathcal{F} \to \mathcal{F}^{+}$ of presheaves, satisfying the following universal property. For every sheaf $\mathcal{G}$ and morphism $\mathcal{F} \to \mathcal{G}$ of presheaves, there is a unique morphism of sheaves $\mathcal{F}^{+} \to \mathcal{G}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}^{+} \\ & \searrow & \downarrow{\scriptstyle \exists !} \\ & & \mathcal{G}. \end{array}$$

Exercise. *The sheafification is unique up to canonical isomorphism.*

Remark. One can equivalently describe this universal property by saying that $\mathrm{Hom}_{\mathrm{Sh}}(\mathcal{F}^{+}, \mathcal{G}) \to \mathrm{Hom}_{\mathrm{PreSh}}(\mathcal{F}, \mathcal{G})$ is a bijection. That is to say, the functor $\mathrm{PreSh}(X) \to \mathrm{Sh}(X)$ taking $\mathcal{F}$ to $\mathcal{F}^{+}$ is left adjoint to the inclusion $\mathrm{Sh}(X) \hookrightarrow \mathrm{PreSh}(X)$.

Definition 86. We construct $\mathcal{F}^{+}$ as a subsheaf of $\mathcal{F}^{\#}$:

$$\mathcal{F}^{+}(U) := \{ s \in \mathcal{F}^{\#} \mid \forall x \in U, \exists x \in U' \subseteq U \text{ and } f \in \mathcal{F}(U') \text{ s.t. } s(x') = f_{x'} \ \forall x' \in U' \}.$$

Proposition 87. *The following facts hold.*

1. *With the obvious restrictions (i.e. those given by $\mathcal{F}^{\#}$), $\mathcal{F}^{+}$ is a sheaf.*
2. *The canonical morphism $\mathcal{F} \to \mathcal{F}^{+}$ is an isomorphism if and only if $\mathcal{F}$ is a sheaf.*
3. *The sheaf $\mathcal{F}^{+}$ satisfies the universal property of the sheafification of $\mathcal{F}$.*

Proof. Important exercise. □

Remark. An alternative/historical/psychological view of sheafification: Set $F := \coprod_{x \in X} \mathcal{F}_x$. This set has an obvious projection $\pi : F \to X$. We topologize F with the basis given by sets $U(f) = \{ f_x \mid x \in U \}$ for each $U \subseteq X$ open and each $f \in \mathcal{F}(U)$. This topological space is called the *espace étale* of $\mathcal{F}$.

Exercise. *The projection $\pi : F \to X$ is a local homeomorphism for the given topology on F.*

The algebraic operations on the stalks $\mathcal{F}_x$ give continuous maps on F, e.g. addition $F \times_X F \xrightarrow{+} F$. Then

$$\mathcal{F}^{+}(U) = \{ \text{continuous sections of } \pi \text{ over } U \}.$$

This was historically the definition of sheaves: as sheaves-of-sections of étale spaces.

15. Thursday, February 28th

Notes taken by Swaraj Pande.

We begin with some remarks about **Sheafification** denoted by $\mathcal{F} \to \mathcal{F}^{+}$. This is functorial: If $\mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, then we have (by the universal property) a unique map $\mathcal{F}^{+} \to \mathcal{G}^{+}$.

Example 88. If A is an abelian group, and $\mathcal{F}$ is the constant presheaf A on X (i.e. $\mathcal{F}(U) = A$ for all U), then $\mathcal{F}^+$ is the sheaf of locally-constant A-valued functions on X.

As in Example 65, if $\mathcal{J} \subset \mathcal{O}_X$ is an ideal sheaf, then $U \mapsto \mathcal{O}_X(U)/\mathcal{J}(U)$ is a presheaf that is usually not a sheaf. We write $\mathcal{O}_X/\mathcal{J}$ for the associated sheaf.

Exercise. *For $\iota : Z \hookrightarrow X$ a closed subset of X, $\mathcal{J} = \mathcal{J}_Z$, then $\mathcal{O}_X/\mathcal{J} \cong \iota_*(\mathcal{O}_Z)$.*

If $\mathcal{F}$ is a subsheaf of $\mathcal{G}$ (of abelian groups), then the quotient sheaf $\mathcal{G}/\mathcal{F}$ is the associated sheaf to the presheaf $U \mapsto \mathcal{G}(U)/\mathcal{F}(U)$. More generally, for a morphism $\varphi : \mathcal{F} \to \mathcal{G}$, of sheaves of abelian groups, we have the sheaf $ker(\varphi)$ which is a subsheaf of $\mathcal{F}$. We define $\mathrm{Im}(\varphi)$ to be the sheaf associated to the presheaf $U \mapsto \mathrm{Im}(\varphi(U))$ and $\mathrm{coker}(\varphi)$ to be the sheaf associated to the presheaf $U \mapsto \mathcal{G}(U)/\mathrm{Im}(\varphi(U))$.

Remark. $\mathrm{coker}(ker(\varphi) \to \mathcal{F}) \cong \mathrm{Im}(\varphi) \cong ker(\mathcal{G} \to \mathrm{coker}(\varphi))$. We also have $\mathcal{F} \oplus \mathcal{G}$ and $\mathcal{F} \times \mathcal{G}$ defined by $U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$ and $U \mapsto \mathcal{F}(U) \times \mathcal{G}(U)$ respectively. Moreover, $\mathcal{F} \oplus \mathcal{G} \cong \mathcal{F} \times \mathcal{G}$. So, **sheaves of abelian groups form an abelian category.**

Remark. (1) If $\mathcal{F}_i$ for $i \in I$ are sheaves of abelian groups, then we have $\bigoplus_{i \in I} \mathcal{F}_i$ satisfying the universal property

$$Hom(\bigoplus \mathcal{F}_i, \mathcal{G}) = \prod Hom(\mathcal{F}_i, \mathcal{G})$$

for any sheaf $\mathcal{G}$. This $\oplus \mathcal{F}_i$ is the associated sheaf to the presheaf $U \mapsto \bigoplus \mathcal{F}_i(U)$. Similarly, we can construct $\varinjlim \mathcal{F}_i$ over I, a poset. Note that here, $(\bigoplus \mathcal{F}_i)_x = \bigoplus (\mathcal{F}_i)_x$.

(2) For $\mathcal{F}_i$ sheaves, $\big(\prod_{i \in I} \mathcal{F}_i\big)(U) = \prod_i \mathcal{F}_i(U)$ is a sheaf. But, $\big(\prod \mathcal{F}_i\big)_x \neq \prod (\mathcal{F}_i)_x$.

Definition 89. The sequence $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ of sheaves of abelian groups is exact at $\mathcal{G}$ if $\psi \circ \varphi = 0$ and the canonical map $\mathrm{Im}(\varphi) \to ker(\psi)$ is an isomorphism.

Exercise. *If $\psi \circ \varphi = 0$, then the sequence is exact at $\mathcal{G}$ if and only if the sequence $\mathcal{F}_x \xrightarrow{\varphi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x$ is exact at $\mathcal{G}_x$ for each x.*

Definition 90. The sequence $0 \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \longrightarrow 0$ is exact if it is exact at $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$. Equivaently, the sequence is exact if and only if

$$0 \longrightarrow \mathcal{F}_x \xrightarrow{\varphi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x \longrightarrow 0$$

is exact at each x.

We note that if $0 \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \longrightarrow 0$ is exact, then for any open subset U of X, $0 \longrightarrow \mathcal{F}(U) \xrightarrow{\varphi(U)} \mathcal{G}(U) \xrightarrow{\psi(U)} \mathcal{H}(U) \longrightarrow 0$ is only left-exact and not necessarily right-exact. So, evaluating the sheaf on an open set is a left-exact functor. Later, when we discuss sheaf cohomology, we will consider the derived functors of this left-exact functor.

Locally ringed spaces and Schemes

Definition 91. A **ringed space** is a topological space X with a sheaf of rings $\mathcal{O}_X$. A **locally ringed space** is a ringed space $(X, \mathcal{O}_X)$ such that each stalk $\mathcal{O}_x X = \varinjlim_{x \in U} \mathcal{O}_X(U)$ is a local ring. A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a function $f : X \to Y$ that is continuous and a map $f^\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$ of sheaves ($f_*(\mathcal{O}_X)$ is the sheaf $U \mapsto \mathcal{O}_X(f^{-1}(U))$ for $U \subset Y$). A morphism of locally ringed spaces (when both X and Y are locally ringed spaces) is a morphism of ringed spaces such that the induced map on the stalks $\mathcal{O}_y(Y) \to \mathcal{O}_x(X)$ maps the maximal ideal of $\mathcal{O}_y(Y)$ into the maximal ideal of $\mathcal{O}_x(X)$ (whenever $f(x) = y$).

Exercise. *We can glue locally ringed spaces along open sets!*

Example 92. Let A be a commutative ring, $X = Spec(A) = \{primeidealsofA\}$ with the Zariski Topology given by: closed sets are exactly of the form $V(I) = \{\mathfrak{p} : \mathfrak{p}$ prime ideal of A such that $\mathfrak{p} \supset I\}$ for I any ideal of A. We often denote an element of the set X by x and the corresponding prime ideal by $\mathfrak{p}_x$.

Define $\mathcal{O}_X(U) = \{s : U \to \coprod_{x \in U} A_{\mathfrak{p}_x} : s(x) \in A_{\mathfrak{p}_x}$ and for all $x \in U$, there is neighbourhood $U' \subset U$ and $f, g \in A$, $g \notin \mathfrak{p}_{x'}$ for any $x' \in U'$ such that $s(x') = \frac{f}{g} \in A_{\mathfrak{p}_{x'}}$ for all $x' \in U'$. Alternatively, we can assign for the basis of open sets given by $X_f = \mathcal{O}_X(X_f)$ the ring A_f. This gives a sheaf on X.

Exercise. *This forms a sheaf of rings denoted by $\mathcal{O}_X$ with the stalk at x given by $\mathcal{O}_x(X) = A_{\mathfrak{p}_x}$.*

Proposition 93. *The canonical map $A \to \mathcal{O}_X(X)$ is an isomorphism. Also, $A_f \to \mathcal{O}_X(X_f)$ is also an isomorphism.*

Proof. Exercise. ☐

For any homomorphism $\varphi : A \to B$ of rings, we get a map of locally ringed spaces $\varphi^* : Spec(B) \to Spec(A)$. On sets,this map is defined by: if $\mathfrak{p}$ is a prime in B, then $\varphi^*(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$ (Note that this map might not take maximal ideals to maximal ideals). In fact,

$$Hom_{rings}(A, B) = Hom_{l.r.s.}(Spec(B), Spec(A))$$

where l.r.s. stands for locally ringed spaces.

Definition 94. A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ that is locally isomorphic to $Spec(A)$ for some ring A i.e. every point $x \in X$ has a neighbourhood U_x and a ring A_x such that $\mathcal{O}_X|_{U_x} \cong Spec(A_x)$. Morphisms of schemes are just morphisms of locally ringed spaces.

Example 95. For toric varieties, X_Δ or $X(\Sigma)$, we can define these over $\mathbb{Z}$ by considering $U_\sigma = Spec(\mathbb{Z}[\sigma^\vee \cap M])$ and gluing these together. We get a scheme over $\mathbb{Z}$ i.e. comes with a map $X(\Sigma) \to Spec(\mathbb{Z})$.

We note that schemes form a category and in fact, they are a full subcategory of the category of locally ringed spaces.

Example 96. A **Closed Subscheme** $i : Z \hookrightarrow X$ is defined by any ideal sheaf $\mathcal{J} \subset \mathcal{O}_X$. In fact, $\mathcal{O}_Z$ is the sheaf on Z with $i_*(\mathcal{O}_Z) = \mathcal{O}_X/\mathcal{J}$.

Example 97. Let $f : X \to Y$ be a morphism of varieties (or schemes) and Z be a closed subvariety (or subscheme), then we have $f^{-1}(Z) \subset X$ is a closed subvariety (or a subscheme): If $Z \subset Y$ is defined by an ideal sheaf $\mathcal{J} \subset \mathcal{O}_Y$, then $f^{-1}(Z)$ is defined by the ideal sheaf in $\mathcal{O}_X$ generated by $f^*(\mathcal{J})$ of not necessarily radical ideals.

$$\begin{array}{ccc} f^{-1}(Z) & \hookrightarrow & X \\ \downarrow & & \downarrow \\ Z & \hookrightarrow & Y \end{array}$$

but $f^{-1}(Z)$ is not the fiber product in the category of schemes.

Example 98. For any maps $X \to Z$ and $Y \to Z$ of schemes, there is a fiber product

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

In fact, locally, we have if $X = Spec(A)$, $Y = Spec(B)$ and $Z = Spec(C)$ then, $X \times_Z Y = Spec(A \otimes_C B)$. And in general, we get $X \times_Z Y$ by gluing these together.

Example 99. Let $S = \bigoplus_{n\geq} S_n$ is an $\mathbb{N}$-graded ring, then we can construct $Proj(S)$: $Proj(S) = \{\text{homogeneous prime ideals } \mathfrak{p}, \mathfrak{p} \not\supseteq S_+\}$. This is covered by $Spec(S_{(F)})$ for $F \in S$ homogeneous elements.

Example 100. If $(X, \mathcal{O}_X)$ is locally $Spec(A_\alpha)$ with A_α finitely generated $k-algebras$ (k a field), then we have the space $X_{cl} \subset X$ of closed points of X coming from maximal ideals in A_α (this is because A_α's are finitely generated over a field) with a one-to-one correspondence between open sets and $\mathcal{O}_{X_{cl}}(U) = \mathcal{O}_X(U)$.

16. Tuesday, March 12th

Written by Carsten Peterson

Constructions depend on the category that we're working in. For example, compare the category of presheaves of abelian groups and sheaves of abelian groups. In these two categories, kernels agree but cokernels disagree.

If $f : X \to Y$ is a map of topological spaces, and $\mathcal{E}$ is a sheaf of abelian groups on X, then $f_*(\mathcal{E})$ is a sheaf of abelian groups on Y:

$$f_*(\mathcal{E})(U) = \mathcal{E}(f^{-1}(U))$$

This is the **direct image** sheaf.

We now wish to construct the inverse image sheaf "$f^{-1}\mathcal{F}$ where $\mathcal{F}$ is a sheaf of abelian groups on Y. It will satisfy the following adjoint relationship:

$$\mathrm{Hom}(f^{-1}(\mathcal{F}), \mathcal{E}) = \mathrm{Hom}(\mathcal{F}, f_*(\mathcal{E}))$$

where the Hom on the left hand side is understood to take place in the category of sheaves of abelian groups on X, and the Hom on the right hand side in the category of sheaves of abelian groups on Y.

Definition 101. We define $f^{-1}(\mathcal{F})$ by:

$$f^{-1}(\mathcal{F})(U) := \{s = (s_x),\ \text{where } s : U \to \textstyle\coprod_{y \in Y} \mathcal{F}_y \text{ and } s(x) \in \mathcal{F}_{f(x)}\}$$

subject to the condition that for all $x \in U$, there are neighborhoods $x \in U'$, $f(x) \in V' \subset Y$ with $f(U') \subset V'$ and a $g \in \mathcal{F}(V')$ so $s(x') = g_{f(x')}$ for all $x' \in U'$

Exercise. *This construction yields a sheaf.*

Exercise. *If $F \to Y$ is the espace etale of $\mathcal{F}$, then elements of $f^{-1}(\mathcal{F})(U)$ correspond to maps:*

$$\begin{array}{ccc} & \nearrow & F \\ & & \downarrow \\ U & \longrightarrow & Y \end{array}$$

i.e. we have the diagram

$$\begin{array}{ccc} X \times_Y F & \longrightarrow & F \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

and $X \times_Y F$ is the espace etale for $f^{-1}(\mathcal{F})$.

Exercise. *The above construction is functorial. That is:*

$$\mathcal{E}_1 \to \mathcal{E}_2 \text{ on } X \ \rightsquigarrow f_*(\mathcal{E}_1 \to f_*(\mathcal{E}_2)$$
$$\mathcal{F}_1 \to \mathcal{F}_2 \text{ on } Y \ \rightsquigarrow f^{-1}(\mathcal{F}_1) \to f^{-1}(\mathcal{F}_2).$$

Exercise. *Given $\mathcal{F}$ on Y, there exists a canonical map $\mathcal{F} \to f_*(f^{-1}(\mathcal{F}))$:*

$$\mathcal{F}(U) \to f^{-1}(\mathcal{F})(f^{-1}(U))$$

with the following property: if $\mathcal{F} \to f_*(\mathcal{E})$ is any homomorphism on Y, there is a unique homomorphism $f^{-1}(\mathcal{F}) \to \mathcal{E}$ such that $\mathcal{F} \to f_*(f^{-1}(\mathcal{F}) \to f_*(\mathcal{E})$ is the given map.

Exercise. *There is a canonical homomorphism $f^{-1}(f_*(\mathcal{E})) \to \mathcal{E}$ for any $\mathcal{E}$ on X so that for any $\mathcal{F}$ and $f^{-1}(\mathcal{F}) \to \mathcal{E}$, there is a unique homomorphism $\mathcal{F} \to f_*(\mathcal{E})$ so that this homomorphism is $f^{-1}(\mathcal{F}) \to f^{-1}(f_*(\mathcal{E})) \to \mathcal{E}$.*

Note that the previous exercises are exactly the unpackaging of the previously mentioned adjoint relationship between f_* and f^{-1}.

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves of $\mathcal{O}_X$-modules (in a ringed space $(X, \mathcal{O}_X)$), we have a presheaf "$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$" given by $U \to \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. We sheafify it to get $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.

We also have a sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$:

$$\mathcal{H}om_{\mathcal{O}_x}(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

This forms a sheaf.

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) = \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})(X)$$

Exercise.

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G},\mathcal{H}) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G},\mathcal{H}))$$
$$\big(\text{Recall: } \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G},\mathcal{H}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G},\mathcal{H}))\big).$$

Note: if $f : X \to Y$ is a map of ringed spaces, $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$ modules, then $f^{-1}(\mathcal{F}$ may not be a sheaf of $\mathcal{O}_X$ modules.

Exercise. *Suppose we have* $X \xrightarrow{f} \mathrm{Specm}(k) = Y$. *Then* $f^{-1}(\mathcal{O}_Y) = k_X$, *i.e. locally constant* k*-valued functions.*

However, $f^{-1}(\mathcal{F})$ is a sheaf of $f^{-1}(\mathcal{O}_y)$-modules.

Definition 102. We define $f^*(\mathcal{F}) \coloneqq \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{F})$. This is a sheaf of $\mathcal{O}_X$-modules.

Proposition 103.

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{F},\mathcal{E}) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{F}, f_*\mathcal{E}).$$

Exercise.

$$f^*\mathcal{O}_Y = \mathcal{O}_X.$$

Exercise. *If* F *is a vector bundle on* Y, *and* $\mathcal{F}$ *is its sheaf of sections,* $\mathcal{F}(U) = \Gamma(U,F)$ *(which is a sheaf of* $\mathcal{O}_X$*-modules), then* $f^*(\mathcal{F})$ *is the sheaf of sections of* f^*F.

Exercise. *If* $i : X \hookrightarrow Y$ *is a locally closed immersion, then* $i^*\mathcal{F} = \mathcal{F}|_X$.

Exercise. *If* E, F *are vector bundles on* X, *and* $\mathcal{E}$, $\mathcal{F}$ *are their corresponding sheaves of sections, then* $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}) = \mathrm{Hom}_{\text{v.b.}}(E,F)$.

Exercise. *Suppose* D *is an effective Cartier divisor, with* $\nVdash \to O(D)$, $Z(s) = D$ *the corresponding section of its line bundle. We then have* $\mathcal{O}_x \to \mathcal{O}_X(D)$ *which is a subsheaf. We have an exact sequence:*

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X/\mathcal{I}) \to 0$$

where $\mathcal{I} = \mathcal{O}(-D) \subset \mathcal{O}_X$ *is the corresponding ideal sheaf. (Note,* $\mathcal{I}/\mathcal{I}^2$ *is sometimes called the conormal sheaf).*

We now move towards cohomology. Suppose X is a complex manifold, say a Riemann surface. We have a short exact sequence of sheaves of abelian groups:

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \xrightarrow{\phi} \mathcal{O}_X^* \to 0$$

where $\phi(f) = e^{2\pi i f}$. We have an exact sequence:

$$0 \to \mathbb{Z}_X(X0 \to \mathcal{O}_X(X) \to \mathcal{O}_X^*(X)$$

where the last arrow is not necessarily a surjection.

Given $g \in \mathcal{O}_X^*(X)$, we can cover X by open sets $\mathcal{U} = (U_\alpha)$ and find $f_\alpha \in \mathcal{O}_X(U_\alpha)$ such that $\phi(f_\alpha) = g|_{U_\alpha}$. On $U_{\alpha\beta} = U_\alpha \cap U_\beta$, $f_\beta - f_\alpha = n_{\alpha\beta}$ where $n_{\alpha\beta}$ is a locally constant integer on $U_{\alpha\beta}$. These satisfy the **cocycle condition**: $n_{\alpha\gamma} = n_{\alpha\beta} + n_{\beta\gamma}$ on $U_{\alpha\beta\gamma}$ for all α, β, γ. (Note that this implies that $n_{\beta\alpha} = -n_{\alpha\beta}$ and $n_{\alpha\alpha} = 0$).

This $n_{\alpha\beta}$ forms a 1-cocycle for $\mathcal{U}$ $(Z^1(\mathcal{U}, \mathbb{Z}_X)$.

If we choose other f'_α with $\phi(f'_\alpha) = g|_{U_\alpha}$, we get $n'_{\alpha\beta} = f'_\beta - f'_\alpha$ which is another cocycle. But $f'_\alpha = f_\alpha + m_\alpha$ where m_α is integer valued on U_α.

$$n'_{\alpha\beta} = (f_\beta + m_\beta) - (f_\alpha + m_\alpha) = n_{\alpha\beta} + (m_\beta - m_\alpha)$$

The 1-cocycles of the form $m_\beta - m_\alpha$ are 1-coboundaries which form a subgroup $B^1(\mathcal{U}, \mathbb{Z}_X) \subset Z^1(\mathcal{U}, \mathbb{Z}_X)$.

We thus get an element in $H^1(\mathcal{U}, \mathbb{Z}_X) := Z^1(\mathcal{U}, \mathbb{Z}_X)/B^1(\mathcal{U}, \mathbb{Z}_X)$.

Exercise. *An element is of the form $g = \phi(f)$ if and only if the corresponding class is 0 in H^1.*

To think about for next time:

$$H^1(\mathcal{U}, \mathcal{O}_X^*) = Z^1(\mathcal{U}, \mathcal{O}_X^*)/B^1(\mathcal{U}, \mathcal{O}_X^*) = \{\text{line bundles on } X \text{ trivialized on } \mathcal{U}\}/\cong$$

We have a map $H^1(\mathcal{U}, \mathbb{O}_X^*) \to H^2(\mathcal{U}, \mathbb{Z}_X)$ (which is equal to $H^2(X, \mathbb{Z})$ is $\mathcal{U}$ is "nice") using "$n_{\alpha\beta\gamma}$". This corresponds to the first Chern class.

17. Thursday, March 14th

18. Tuesday, March 19th

19. Thursday, March 21st